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BY

GEORGE W. PARKER, M.A.
OF TRINITY COLLEGE, DUBLIN

SECOND EDITION
REVISED AND ENLARGED

LONGMANS, GREEN, AND CO.
39, PATERNOSTER ROW, LONDON
NEW YORK, BOMBAY, AND CALCUTTA
1921

PHYSICS DEPT.

PRINTED AT THE
DUBLIN UNIVERSITY PRESS
BY PONSONBY & GIBBS.

PREFACE.

THE present Work is written for the use of those students whose knowledge of Mathematics is limited to an acquaintance with Elementary Geometry, the solution of Simple Quadratic Equations in Algebra, and a few fundamental propositions in Plane Trigonometry.

The many real and perplexing difficulties which the study of Mechanics presents to the beginner are in a great measure due to the fact that he has not, at the very outset, acquired clearness of ideas about the first principles which form the groundwork of the subject. There is, therefore, some fear lest his work should degenerate into a mere manipulation of symbols and the application of rules, uninteresting and unsuggestive, because little understood. For this reason great care has been taken in explaining and illustrating the nature of the Resolution of Forces and the Principle of Moments in Statics, and the applications of Newton's Second Law of Motion in Dynamics. Also the numerous examples which are worked out are solved, wherever possible, without making use of mathematical formulæ, but by the direct application of first principles.

It is right to mention that many suggestive examples have been borrowed from papers set to Students at the various Universities, and more especially from those proposed at Examinations of Engineering Students in the University of Dublin.

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The older method of treating Statics before Dynamics has been followed, partly because the subject, if taught in this manner, is, in the author's opinion, more easily grasped by the beginner, and also on account of the much greater practical importance of Statics, especially to the mechanical engineer. Those, however, who prefer to base the principles of Statics on Newton's Laws of Motion should, in the first instance, read Chapters XI and XII, also Articles 223-225 and 230-232; and, if this course is followed, the book can be used with equal advantage whichever method is adopted.

In conclusion, the author will feel grateful for any corrections or suggestions from teachers or students with a view to improvements in a future edition.

PREFACE TO THE SECOND EDITION.

In this issue the text has been revised and corrected. The number of examples has been increased. A large amount of matter has been added in connexion with projectiles, oblique collision, impulse, and kinetic energy. In the Appendix will be found an additional proof of Varignon's Theorem of Moments for Concurrent Forces.

The author wishes to thank many friends and correspondents who have suggested most of the improvements.

TRINITY COLLEGE, DUBLIN,
January, 1921.

CONTENTS.

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PART I.—STATICS.

CHAP. PAGE

I. INTRODUCTION, 1
  Measurement of Forces, 3
  Axioms, 6

II. COMPOSITION OF FORCES, 7
  The Parallelogram of Forces, 7
  The Triangle of Forces, 14
  The Polygon of Forces, 17
  Lami's Theorem, 19

III. RESOLUTION OF FORCES, 22
  Components and Resolutes, 23
  Conditions for Equilibrium, 29

IV. MOMENTS, 31
  Varignon's Theorem of Moments, 33

V. PARALLEL FORCES, 39
  Resultant of Like Parallel Forces, 39
  Resultant of Unlike Parallel Forces, 41
  Couples, 47

VI. CENTRE OF GRAVITY, 55
  Centre of a System of Parallel Forces, 55
  Centre of Gravity of a Triangle, 59
  Properties of the Centre of Gravity, 71
  Stable, Unstable, and Neutral Equilibrium, 73

VII. CONDITIONS OF EQUILIBRIUM, 75
  For a number of Coplanar Forces, 81

CHAP. PAGE
VIII. MACHINES: THE LEVER, AND WHEEL AND AXLE, . . 85
Three Orders of Levers, ... 86
The Common Balance, ... 91
The Common or Roman Steelyard, ... 95
The Danish Steelyard, ... 96
The Wheel and Axle, ... 99
The Differential Wheel and Axle, ... 101

IX. THE PULLEY AND INCLINED PLANE, ... 104
The First System of Pulleys, ... 105
The Second System of Pulleys, ... 109
The Third System of Pulleys, ... 111
The Inclined Plane, ... 115
The Wedge, ... 121
The Screw, ... 122

X. FRICTION, ... 126
Laws of Limiting Friction, ... 127
The Coefficient and Angle of Friction, ... 128
The Rough Inclined Plane, ... 129

PART II.—DYNAMICS.

XI. VELOCITY AND ACCELERATION, ... 139
Uniformly Accelerated Motion, ... 141
Falling Bodies, ... 148

XII. LAWS OF MOTION, ... 155
Newton's Laws, ... 155
The Absolute Unit of Force. The Poundal, . . . 158
Weights of Bodies Proportional to their Masses, . . 161

XIII. APPLICATION OF THE LAWS OF MOTION, ... 165
Motion on a Smooth Inclined Plane, ... 165
Motion on a Rough Inclined Plane, ... 167
Motion on a Rough Horizontal Plane, ... 169
Pressure of a Body on a Lift, ... 171

XIV. FURTHER APPLICATIONS OF THE LAWS OF MOTION, . . 174
Motion of Two Masses connected by a String, . . 174
Atwood's Machine, ... 180

CHAP. PAGE
XV. COMPOSITION OF VELOCITIES AND ACCELERATIONS, . . 187
The Parallelogram of Velocities, ... 188
The Triangle and Polygon of Velocities, . . . 191
Relative Velocity, ... 192
The Parallelogram of Accelerations, ... 194

XVI. UNIFORM MOTION IN A CIRCLE. PROJECTILES, . . 197
Centripetal Force, ... 198
Projectiles, ... 204

XVII. WORK AND ENERGY, ... 218
The Principle of Work, ... 221
Proof of Formula for the Screw from the Principle of Work, 222
Impulse, ... 226
Potential and Kinetic Energy, ... 228
Law of Conservation of Energy, ... 230

XVIII. IMPACT, ... 231
Direct Impact of two Spheres, ... 233
Loss of Kinetic Energy, ... 234
Total Impulse, ... 235
Periods of Compression and Restitution, . . . 236
Oblique Impact, ... 239

XIX. SIMPLE HARMONIC MOTION: THE PENDULUM, . . 249
Harmonic Motion, ... 249
The Pendulum, ... 252
To find the value of *g*, ... 254
The Conical Pendulum, ... 255

APPENDIX, ... 258

ANSWERS, ... 260

PART I.

STATICS.

CHAPTER I.

INTRODUCTION.

1. **Mechanics.**—The science which treats of the action of forces on bodies is called *Mechanics*.

This subject is generally divided into two branches.

(1) **Statics**, which deals with the laws to which forces are subject when the bodies on which they act are in a state of rest.

(2) **Dynamics**, which treats of forces which by their action on bodies produce motion.

Some writers include Statics in Dynamics; and therefore if taken in this sense, the term "Dynamics" might be used instead of what is generally called Mechanics.

2. **Matter** is something (we know not what) of whose existence we form an intuitive judgment, from the fact that the mind is conscious of such qualities as solidity, extension, inertia, &c.

3. A **Body** is a limited portion of matter; and a body whose size or bulk is so small that its mass may practically be regarded as concentrated into a mathematical point is called a **Particle.**

4. A **Rigid Body** is such that its parts always preserve the same relative positions with respect to one another when external forces are applied to it; or, in other words, there is no change produced in its shape by the action of external forces.

There is no body existing in nature which is absolutely rigid, for even in the case of a block of iron or stone there is a slight strain or change in shape produced by the action of external forces; but these changes are, under ordinary circumstances, so small that, for present purposes, we may regard them as non-existent.

5. **Mass.**—The quantity of matter in a body is called its *mass*. In the British Isles, the unit of mass is the **pound**, the **standard pound mass** being that of a lump of platinum preserved in the Exchequer Office, London, several exact copies of which are kept in various places, so that if the original were lost it might be restored.

In France, the unit of mass is the **gramme** or **gram**, a standard kilogramme or 1000 grammes being carefully preserved, as in the case of the English pound.

6. **Force.**—When we raise a weight off the ground or commence to wheel a truck which has previously been at rest, we are conscious of exerting a muscular effort to produce the motion; or, if the body be already moving, we find that a similar effort has to be made either to bring it to rest or to produce any other alteration in its motion. Our first conception of force is thus obtained from muscular effort exerted by a living agent.

Living agents, such as men and brute beasts, have at all times been used to supply force; and in the early stages of civilization they formed the chief sources of energy by means of which work could be performed. Gradually, however, the human race learned that natural agents other than living creatures can be utilized to produce the same or even greater results: thus, when water falling from a height turns the wheel of a water-mill, the force of gravitation is made use of to perform work. Similarly, the force exerted by the air in motion on the sails of a wind-mill is used to work the mill; and the same remarks apply to steam, electricity, &c.

7. **Definition.**—A **Force** *is anything which tends to change the state of rest or uniform motion of a body.*

Measurement of Forces.

8. In order to measure a force, we must have (1) some test to apply to determine when two forces are equal; (2) some unit of measurement with which to compare other forces.

9. **Equal Forces.**—Two forces are said to be equal if, when they are applied to a particle in opposite directions, the particle remains at rest.

From this it follows that, when it is said that the magnitude of one force is twice that of another, it is meant that the first, when applied to a particle in any direction, will exactly balance or equilibrate two forces, each equal to the second, acting on the same particle in the opposite direction. Similarly, we can get a clear idea of what is meant by a force whose magnitude is three, four, or five times that of another force; and hence, when some definite unit of force is laid down, the magnitude of any other force can be expressed in terms of that unit.

10. **Weight.**—The Earth attracts the mass of every body towards its centre with a force which is called the *weight* of the body. The weight of a body is slightly different at different parts of the Earth; for, on account of the Earth being slightly flattened at the poles,* a body near the poles is closer to the centre of the Earth than when near the equator, and therefore the Earth attracts it with a greater force; and hence the weight of a body increases slightly as we approach the poles.

We shall see subsequently that the force with which the Earth attracts a body is directly proportional to the mass of the body, and hence that the weight of a body is a measure of its mass.

---

* The polar diameter of the Earth is 26 miles shorter than the equatorial ("Elements of Astronomy," Art. 19).

11. **Measurement of Forces.**—The *statical unit of force* is a force equal to that exerted by the attraction of the Earth on a mass of one pound. This is called "a force of one pound," which really means "a force equal to the weight of one pound."

12. **Forces may be represented by straight lines.**—In order to completely determine a force, we must know

(1) Its point of application; i.e. the point of the body at which the force is applied.
(2) Its direction.
(3) Its magnitude.

These three conditions can be represented by a straight line; thus, when it is said that the line AB represents a force acting on a body, it is meant (1) that A is the point in the body at which the force is applied; (2) the direction from A to B as indicated by the arrow-head is the direction in which the force acts; and (3) if there are as many units of length in AB as there are pounds weight in the force, then AB also represents the force in magnitude.

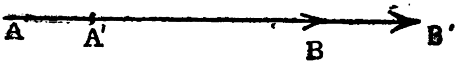

FIG. 1.

The direction of the force is very conveniently represented by an arrow-head, or it may be indicated by the order of the letters used: thus AB represents a force acting from A to B, and BA a force acting in the reverse direction, viz. from B to A.

13. **Action and Reaction.**—When one body presses against another the latter body reacts against the former with an equal and opposite force. Thus, when a book rests on a table, the table exerts an upward force on the book equal and opposite to that exerted by the book on the table. That Action and Reaction are equal and opposite is included in Newton's Third Law of Motion.

It is very important that the student should, at the outset, clearly realize this duality of forces, viz. that every action is accompanied by an equal and opposite reaction, and that therefore force cannot exist independent of matter.

14. **Perfectly Smooth Surfaces.**—When two bodies with perfectly smooth surfaces are in contact, the reaction between them is normal or perpendicular to their common surface. That this must be the case can be seen if we place a polished slab of wood or metal on very smooth ice: it will be found that very little resistance is experienced in moving it along the ice. The fact that any resistance, no matter how small, is experienced is due to the fact that there is no body whose surface is *perfectly* smooth; from this we conclude that if the surfaces of the ice and slab were perfectly smooth there could be no force between them parallel to their common surface, and that, therefore, the only reaction between two such bodies is perpendicular or normal to that surface.

15. **Tension.**—When a force is applied to a body by means of a string, it is called *tension.* Thus, if we tie one end of a light string (whose weight may be neglected) to a body, and, with the hand, pull the other end with a force of 10 lbs. wt., then the body is pulled by the string with a force of 10 lbs. wt., and there is also an equal and opposite force exerted by the string on the hand (Art. 13); the string is then said to have a tension of 10 lbs. wt. throughout its length. It is also evident that this tension is not altered if the string be passed round one or more *smooth* pegs: that is, *the tension of a light string passing round one or more smooth pegs is the same throughout its length.*

16. **Definition.**—When a number of forces acting on a body keep the body at rest, the forces are said to be in **Equilibrium.**

It is necessary in Mechanics to make the following assumptions, which may be regarded as axioms:—

17. AXIOM 1. **Transmissibility of a Force.**—If a force acts on a rigid body, its point of application can be transferred to any other point in the body in the line of action of the force without altering its effect.

Thus a force represented by AB acting on a rigid body (fig. 1) will be unaltered in effect if its point of application be changed from A to any other point A' in the body situated on AB; that is, if $A'B'$ be cut off equal in length to AB, then a force represented by $A'B'$ is in every way identical with one represented by AB.

18. AXIOM 2. If a system of forces acts on a body, any other system which in itself would keep the body in equilibrium can be introduced or removed without any alteration in the system.

As an example of this axiom, we often introduce two equal and opposite forces into a system, or remove two equal and opposite forces from a system, without any alteration of effect.

EXAMPLES I.

1. If a force of 20 lbs. wt. be represented by a line 2″ long, how long will be the line which represents a force of 1 ton?

2. If a force of P lbs. wt. be represented by a line a inches long, what force is represented by a line b inches long?

3. If a force of P lbs. wt. be represented by a line a inches long, what length of a line would represent a force of Q lbs. wt.?

4. If a line 6″ long represents a force of 48 lbs. wt., what force will be represented by a line of length 2′ 9″?

CHAPTER II.

COMPOSITION OF FORCES.

19. In this Chapter we shall explain how a number of forces may be compounded into a single force.

Definition : Resultant.—When a number of forces acting on a body can be balanced by a single force, a force equal and opposite to this latter force is called the *Resultant* of the other forces.

It follows from this that the resultant of a number of forces can produce the same amount of motion in a body as these forces combined.

Evidently, the resultant of two forces whose line of action is in the same straight line and in the same direction is equal in magnitude to their sum, and acts in the same straight line and in the same direction as the two forces; and if the lines of action of the forces are in opposite directions, their resultant is equal to their difference, and acts in the direction of the greater of the two forces.

The Parallelogram of Forces.

20. **If two forces, acting on a particle, be represented in magnitude and direction by two sides of a parallelogram drawn from their point of application, their résultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.**

Experimental Proof of the Parallelogram of Forces.

(*a*) Take three strings and knot them together at a point (fig. 2). At the extremities of the strings attach any three weights *P*, *Q*, and *R* respectively (any two of which taken together are greater than the third*). Pass the strings carrying the weights *P*, *Q*, and *R* over three smooth pulleys *L*, *M*, and *N* respectively, fixed in front of a vertical blackboard. The point *O* where the strings are knotted will now move until the whole system is in equilibrium, as represented in the figure.

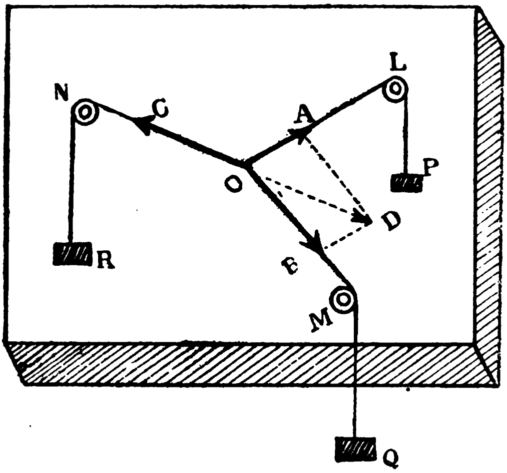

FIG. 2.

(*b*) *Construction.*—Now measure off, *on the blackboard* *OA*, *OB*, and *OC* containing as many units of length (say, inches) as there are units of weight (say, pounds) in *P*, *Q*, and *R* respectively; then complete the parallelogram *OD*, and join *OD*.

(*c*) It is now invariably found that the diagonal *OD* is equal in magnitude and opposite in direction to *OC*.

Now the knot at *O* is in equilibrium under the action of the three forces *P*, *Q*, and *R* represented by *OA*, *OB*, and

---

* The reason of this will be seen on reading Art. 23.

OC respectively: hence a force represented in magnitude and direction by the diagonal *OD* is the resultant of the forces represented by *OA* and *OB*.

21. *Corollary.—If two forces be represented in magnitude and direction by two sides OA and OB (fig. 3) of a triangle OAB, their resultant is represented by twice the median OL, where L is the middle point of AB.*

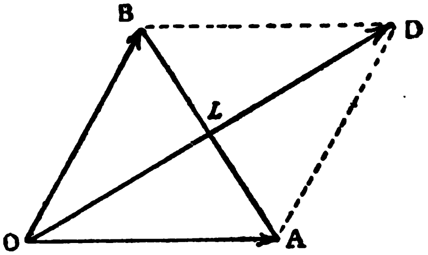

FIG. 3.

For, complete the parallelogram *OD*. Then, since the diagonals of a parallelogram bisect one another, $OD = 2OL$; but *OD* represents the resultant of the two forces: therefore $2OL$ represents the resultant.

22. *To find the magnitude and direction of the resultant of two forces whose directions are at right angles to one another.*

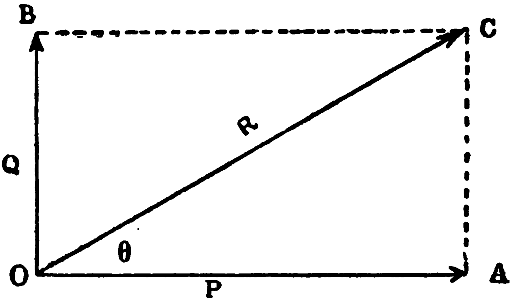

FIG. 4.

Let the forces *P* and *Q* be represented by *OA* and *OB* at right angles to one another. Complete the rectangle *OC*.

Join OC; then the diagonal OC represents the resultant R. Now, by Euclid (I. 47),

$$OC^2 = OA^2 + AC^2 = OA^2 + OB^2;$$

$$\therefore\ R^2 = P^2 + Q^2; \quad \therefore\ R = \sqrt{P^2 + Q^2}.$$

Now, to find the direction of the resultant:—

Let θ be the angle which the direction of R makes with OA. Then we have

$$\tan\theta = \frac{Q}{P};$$

and therefore, if the numerical values of P and Q be given, the value of θ can be found by reference to a book of tables.

Ex. 1. Find, in magnitude and direction, the resultant of forces of 3 lbs. and 4 lbs. wt. acting at right angles.

Here, $R^2 = 3^2 + 4^2 = 25$; $\therefore\ R = 5$ lbs. wt.

Also, if θ be the angle which the line of action of the resultant makes with that of the 3 lbs. force, we have

$$\tan\theta = \frac{4}{3};$$

and we find, on reference to a table of tangents, that this angle is 48° 26′.

Ex. 2. Two forces whose directions are at right angles to one another give a resultant of 13 lbs. wt.; one of the forces is equal to 5 lbs. wt. Find the other force.

Here $R^2 = P^2 + Q^2$, $13^2 = 5^2 + Q^2$, $169 = 25 + Q^2$;

$$\therefore\ Q^2 = 144, \quad \text{or} \quad Q = 12 \text{ lbs. wt.}$$

Ex. 3. Find the magnitude and direction of the resultant of 9 lbs. and 40 lbs. wt. respectively whose directions are at right angles to one another.

Ans. 41 lbs. wt.; and whose direction makes an angle $\tan^{-1}\frac{40}{9}$* with that of 9 lbs.

23. It is obvious, since any two sides of a triangle taken together are greater than the third side, that the resultant of any two forces acting on a particle at either an acute or obtuse angle is less than the sum of the forces and greater than their difference; and that the greatest resultant which can be obtained from two forces is when the lines of action

---

* The symbol $\tan^{-1}\frac{40}{9}$ means *the angle whose tangent is* $\frac{40}{9}$.

of the forces are in the same direction, in which case the resultant is equal to the sum of the forces. Also, the least resultant which two forces can produce is when their lines of action are in opposite directions, in which case the resultant is equal to their difference.

Ex. 1. Find the greatest and least resultants of two forces whose magnitudes are 8 lbs. and 15 lbs. wt. respectively. Also, find what would be the resultant of these forces if their lines of action were at right angles.

Here the greatest resultant $= 15 + 8 = 23$ lbs. wt.,
and the least resultant $= 15 - 8 = 7$ lbs. wt.

Also, when at right angles, $R = \sqrt{8^2 + 15^2} = 17$ lbs. wt.

Ex. 2. If the greatest and least resultants of two forces be 12 lbs. and 2 lbs. respectively, find the magnitudes of the forces.

Here, if P and Q represent the forces (P being the greater),

$$\begin{aligned} P + Q &= 12 \\ P - Q &= 2 \\ \hline 2P &= 14 \end{aligned}$$

$\therefore P = 7$ lbs. wt., and $Q = 5$ lbs. wt.

To find the resultant of two forces acting at any angle α.

24. Let the forces P and Q acting at an angle α be represented by OA and OB respectively (figs. 5, 6); then the diagonal OC represents the resultant R.

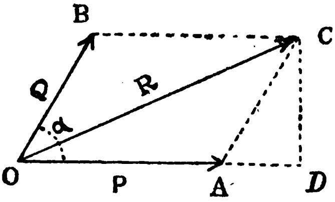

FIG. 5.

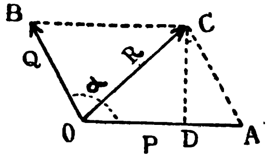

FIG. 6.

Drop CD perpendicular on OA (or OA produced). Now (fig. 5),

$$OD = OA + AD = OA + AC \cos CAD,$$

or $$OD = OA + OB \cos \alpha = P + Q \cos \alpha.$$

[If the angle a be obtuse (fig. 6), the point D will lie between O and A; and we will have

$$OD = OA - DA = OA - AC \cos CAD;$$

but the $\angle\, CAD = 180 - a$; $\therefore\ \cos CAD = -\cos a$;

$\therefore\ OD = OA - AC \times - \cos a = P + Q \cos a$, as before.]

Also, in both cases,

$$CD = AC \sin CAD = Q \sin a;$$

But $$OC^2 = OD^2 + CD^2;$$

$$\therefore\ R^2 = (P + Q \cos a)^2 + Q^2 \sin^2 a$$
$$= P^2 + 2PQ \cos a + Q^2(\sin^2 a + \cos^2 a);$$

but $$\sin^2 a + \cos^2 a = 1;$$

$$\therefore\ R^2 = P^2 + Q^2 + 2PQ \cos a.$$

(This result may also be obtained by making use of Euclid, Bk. II., Props. 12 and 13.)

Also, the direction of the resultant can be found; for, if $\angle\, AOC = \theta$, we have

$$\tan \theta = \frac{CD}{OD} = \frac{Q \sin a}{P + Q \cos a}.$$

Ex. 1. Find the resultant of two forces equal to 7 lbs. and 8 lbs. wt. respectively, and acting at an angle of 60°.

Measure off OA and OB equal to 7 units and 8 units of length respectively, making the angle $AOB = 60°$. Complete the parallelogram OC; then the line OC represents R. Draw CD perpendicular to OA produced.

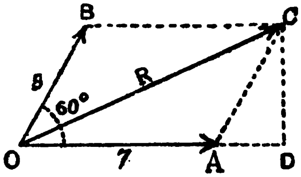

FIG. 7.

Now,

$$CD = AC \sin CAD = 8 \sin 60 = \frac{8\sqrt{3}}{2} = 4\sqrt{3},$$

and $$OD = OA + AD = 7 + 8 \cos 60 = 7 + 4 = 11;$$

but $$OC = \sqrt{OD^2 + CD^2} = \sqrt{121 + 48} = 13.$$

Hence the magnitude of the resultant is 13 lbs. wt.

Also, the direction of the resultant can be found; for, if θ be the angle which it makes with OA, we have

$$\tan \theta = \frac{CD}{OD} = \frac{4\sqrt{3}}{11} = \cdot 6298,$$

from which θ is found to be 32° 48′.

Ex. 2. Find the resultant of forces equal respectively to 7 lbs. and 11 lbs. wt., and inclined at an angle of 120°. The details of the construction are the same as in last example. Here (see fig. 6) OA and OB represent the 7 lbs. and 11 lbs. forces respectively, the angle AOB being 120°.

$$CD = AC \sin 60 = \frac{11\sqrt{3}}{2},$$

$$OD = OA - DA = 7 - 11 \cos 60 = \frac{3}{2},$$

$$OC = \sqrt{\frac{9}{4} + \frac{363}{4}} = \sqrt{93}; \quad \therefore R = \sqrt{93} \text{ lbs. wt.}$$

Also, $$\tan\theta = \frac{11\sqrt{3}}{3} = 6{\cdot}351; \quad \therefore \theta = 81^\circ\, 57'.$$

NOTE.—The magnitudes of the resultants in the last two examples could also have been quickly found by substituting the values of P, Q, and a in the formula

$$R^2 = P^2 + Q^2 + 2PQ \cos a;$$

but the student will become more familiar with principles by working out examples as above.

EXAMPLES II.

1. Find the resultant of two forces of 7 and 24 lbs. wt. respectively which act at right angles.

2. One of two forces which act at right angles to one another is 11 lbs. wt.: their resultant is 61 lbs. wt. Find the magnitude of the other force.

3. Four forces whose magnitudes are respectively 4, 6, 8, and 10 lbs. wt. act on a particle in directions north, south, east, and west respectively: find the magnitude and direction of their resultant.

4. Find the greatest and least resultants of two forces whose magnitudes are 5 lbs. and 3 lbs. wt. respectively. What would be the magnitude of the resultant of these forces if their directions were inclined at an angle of 60°?

5. Two forces, each equal to 10 lbs. wt., act on a particle at an angle of 120°: find the magnitude and direction of their resultant.

6. Two forces, each equal to P, give a resultant equal to P: find the angle between the forces.

7. Two forces whose magnitudes are 2 lbs. and 4 lbs. wt. respectively are inclined to one another at an angle of 120°: find the magnitude and direction of their resultant.

8. Two forces whose magnitudes are in the ratio 3 : 5 give a resultant of 35 lbs. wt. If the angle of inclination of the forces be 60°, find the magnitudes of the forces.

9. Three pegs form an equilateral triangle; if an india-rubber band be stretched round the pegs, and if the tension of the band be equal to a weight of 10 lbs., find the resultant pressure on each peg.

10. Six pegs form a regular hexagon: if an india-rubber band be stretched round the pegs, prove that the resultant pressure on each peg is equal to the tension of the band (see Examples 5 and 6).

11. Two forces act at an angle of 120°; one of the forces is equal to a weight of 1 lb.; the magnitude of the resultant is $\sqrt{3}$ lbs. wt.: find the other force.

12. Three forces whose magnitudes are respectively 1, $3\sqrt{2}$, and 5 lbs. wt. keep a particle in equilibrium: find the angle between the first pair.

13. Three forces whose magnitudes are proportional to $2\sqrt{3}$, $\sqrt{3}$, and 3 act on a particle and keep it in equilibrium: find the angles between the directions of the forces.

14. Find the resultant of two forces of 7 lbs. and 12 lbs. wt. respectively inclined to one another at an angle whose cosine is $\frac{5}{13}$.

15. A number of forces of given magnitude and direction act on a particle: find in what direction an additional force must act in order that the resultant of all should be (1) as great as possible, (2) as small as possible.

16. A number of forces act on a particle and keep it in equilibrium; one of the forces ceases to act: find in what direction the particle will now move.

25. **The Triangle of Forces.**—*If three forces acting on a particle can be represented in magnitude and direction by the sides of a triangle taken in order, they shall be in equilibrium.*

N.B.—What is meant by the sides being *taken in order* is that the directions must be either *CB, BA, AC,* or *BC, CA, AB* (fig. 8), i.e. either all in the same direction

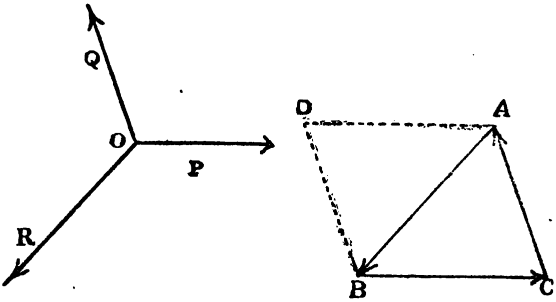

Fig. 8.

as the hands of a clock move or all in the contrary direction.

Let forces P, Q, and R act on a particle at O and ABC a triangle,

P being represented in magnitude and direction by BC,
Q being represented in magnitude and direction by CA,
R being represented in magnitude and direction by AB;

or, as it might be expressed as regards their magnitude,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

Then shall P, Q, and R be in equilibrium.

Complete the parallelogram $BCAD$.

Then BD being equal and parallel to CA will also represent the force Q.

Now the resultant of the forces BC and BD is represented by the diagonal BA; that is the resultant of P and Q acting at O is represented in magnitude and direction by BA.

Hence, since R is represented by AB, the resultant of P, Q, and R acting at O is equal to the resultant of BA and AB, and is therefore zero. Hence P, Q, and R are in equilibrium.

NOTE.—It is to be borne in mind that if the lines of action of the three forces P, Q, and R acting on a body were actually *along the sides* BC, CA, and AB respectively of the triangle, they would not be in equilibrium. In fact, to be in equilibrium they must meet in a point.

Corollary 1.—From this it follows that the resultant of two forces represented in magnitude and direction by BC and CA is represented by AB reversed, that is by BA.

Corollary 2.—It is evident that, if the directions of the forces be either perpendicular or equally inclined to the sides of the triangle to which they are proportional, the forces will still be in equilibrium; for, if they are in equilibrium when their directions are parallel to the sides of the triangle, they will still be in equilibrium when they are turned through a common angle.

26. **Converse of the Triangle of Forces.**—*If three forces acting on a particle be in equilibrium, and any triangle be drawn whose sides are parallel to the directions of the forces, then the lengths of the sides of the triangle are proportional to the magnitudes of the forces.*

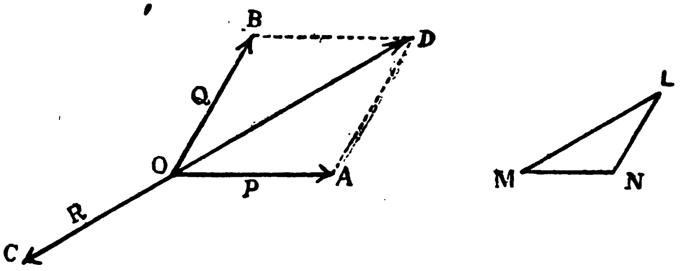

Fig. 9.

Let the three forces P, Q, and R, acting on a particle at O, and represented by OA, OB, and OC respectively, be in equilibrium.

Complete the parallelogram $OADB$, and join OD.

Hence the resultant of P and Q is represented by OD, and therefore, since R must be equal and opposite to the resultant of P and Q, OC and OD are in the same straight line. Also AD is equal and parallel to OB.

Hence the forces P, Q, and R are parallel and proportional to the sides OA, AD, and DO respectively of the triangle OAD taken in order.

Now if any other triangle LMN (fig. 9) be drawn whose sides are parallel to the sides of the triangle OAD, viz. MN to OA, NL to AD, and LM to DO, then, since the two triangles are equiangular, their sides are proportional (Euclid, VI. 4);

$$\therefore\ MN : NL : LM = OA : AD : DO;$$

but

$$P : Q : R = OA : AD : DO :$$

$$\therefore\ P : Q : R = MN : NL : LM;$$

or, as it may otherwise be written,

$$\frac{P}{MN} = \frac{Q}{NL} = \frac{R}{LM}.$$

27. **The Polygon of Forces.**—*If any number of forces, acting on a particle, be represented in magnitude and direction by the sides of a polygon taken in order, they shall be in equilibrium.*

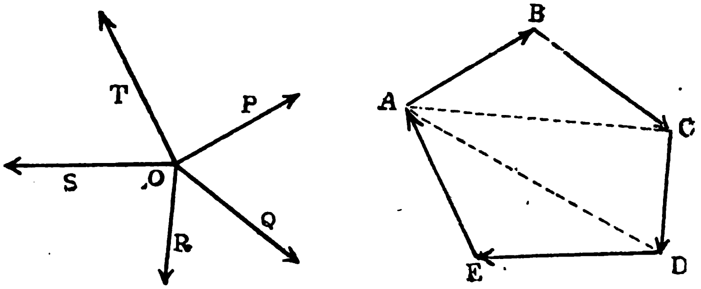

FIG. 10.

Let P, Q, R, S, and T, acting on a particle at O, be represented in magnitude and direction by the sides AB, BC, CD, DE, and EA respectively, of the polygon $ABCDE$; then shall the forces be in equilibrium.

Join AC and AD.

Now the resultant of forces represented by AB and BC is represented by AC (*Cor.* 1, Art. 25).

Again, the resultant of forces AC and CD is represented by AD; and the resultant of forces AD and DE is represented by AE. Hence there are only two forces left, represented by AE and EA respectively, which, being equal and opposite, are in equilibrium. Therefore, P, Q, R, S, and T, acting at O, are in equilibrium.

It is obvious, from the above proofs, that the theorem is true even when the sides of the polygon are not in the same plane.

28. We have seen that the converse of the Triangle of Forces is true; but the converse of the Polygon of Forces is not true—that is, if a polygon be drawn with its sides parallel to the directions of any number of forces which keep a particle in equilibrium, it does not follow that the lengths of these sides are proportional to the magnitudes of the forces.

For, supposing $ABCDE$ to be a polygon whose sides are parallel to the directions of the forces, then, if $A'B'$ be drawn parallel to AB, therefore $A'B'CDE$ is also a polygon whose sides are parallel to the forces: hence, if the converse of the Polygon of Forces be true, these two polygons have their sides proportional to one another;

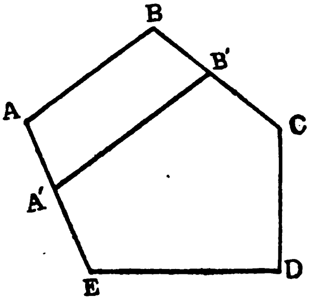

FIG. 11.

$$\therefore DE : EA :: DE : EA'; \quad \therefore EA = EA',$$

which is absurd.

To find Graphically the Resultant of any number of Forces acting on a Particle.

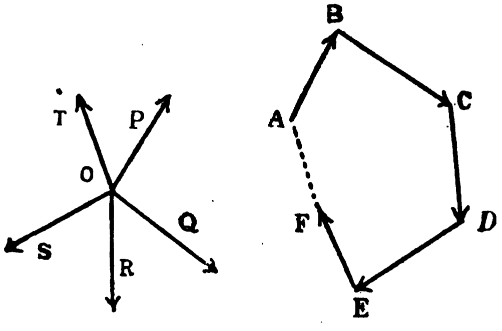

FIG. 12.

29. Let the forces be P, Q, R, S, and T, acting on a particle at O; it is required to find their resultant by a Graphic construction.

Take any point A, and draw AB, BC, CD, DE, and EF to represent P, Q, R, S, and T respectively, in magnitude and direction.

Now, if the last point *F* coincides with the first point *A*, the forces, being parallel and proportional to the sides of a closed polygon taken in order, are in equilibrium, and the resultant is zero.

If *F*, however, does not coincide with *A*, the line that closes the polygon reversed in direction—viz. *AF*—represents the resultant in magnitude and direction. Hence a force acting at *O* represented by *AF* is the resultant. This follows at once from the Polygon of Forces.

30. **Lami's Theorem.**—*If three forces acting on a particle be in equilibrium, each is proportional to the sine of the angle between the other two.*

Let the forces *P*, *Q*, and *R* (fig. 9) be in equilibrium, and let them be represented by *OA*, *OB*, and *OC* respectively. As before, *OD* represents the resultant of *P* and *Q*, and is in the same straight line as *OC*.

Now, we know by Trigonometry that the sides of a triangle are proportional to the sines of the opposite angles; therefore in the triangle *OAD* we have

$$\frac{OA}{\sin ODA} = \frac{AD}{\sin AOD} = \frac{OD}{\sin OAD};$$

but

$$\sin ODA = \sin BOD = \sin(180^\circ - BOC) = \sin BOC,$$

also

$$\sin AOD = \sin(180^\circ - COA) = \sin COA,$$

and

$$\sin OAD = \sin(180^\circ - AOB) = \sin AOB.$$

Also $AD = OB$;

$$\therefore \quad \frac{OA}{\sin BOC} = \frac{OB}{\sin COA} = \frac{OC}{\sin AOB},$$

or

$$\frac{P}{\sin \widehat{QR}} = \frac{Q}{\sin \widehat{RP}} = \frac{R}{\sin \widehat{PQ}},$$

where $\widehat{QR}$ denotes the angle between the directions of the forces *Q* and *R*, etc.

Ex. Three forces acting on a particle are in equilibrium: the angle between the first and second is 120°; the third is at right angles to the second: if the magnitude of the greatest force be 10 lbs. wt., find the magnitudes of the other two.

Here the angles between the directions of the forces are 90°, 150°, and 120°; and since sin 90° is greater than either sin 150° or sin 120°, the greatest force must be opposite the angle 90°. Hence, if Q and R represent the other forces, we have

$$\frac{10}{\sin 90^\circ} = \frac{Q}{\sin 150^\circ} = \frac{R}{\sin 120^\circ},$$

or

$$\frac{10}{1} = \frac{Q}{\frac{1}{2}} = \frac{R}{\frac{\sqrt{3}}{2}};$$

$$\therefore \quad Q = 5 \text{ lbs. wt., and } R = 5\sqrt{3} \text{ lbs. wt.}$$

EXAMPLES III.

1. Three forces of 15, 20, and 25 units respectively act on a particle: if the directions of the forces are parallel to the sides of a triangle whose lengths are 3, 4, and 5 inches respectively, show that the forces are in equilibrium.

2. If, in the last question, the magnitudes of the forces are 15, 20, and 29 units respectively, find the magnitude and direction of their resultant.

3. Forces of 11, 12, and 13 units respectively act on a particle inclined mutually at angles of 120°: find the magnitude of their resultant and its inclination to the force of 12.

4. Three forces act perpendicularly to the sides of a triangle at their middle points, and each is proportional to the side on which it acts: show that they will equilibrate each other.

5. Forces proportional to the sides of any plane polygon acting perpendicularly to these sides at their middle points will equilibrate?

6. Three forces acting on a particle are in equilibrium and are proportional to $\sqrt{3} + 1$, $\sqrt{6}$, and 2: find the angles at which they are inclined to each other.

7. If $ABCD$ be a parallelogram, and forces acting on a particle be represented by AB, BD, and CA respectively, find their resultant; also find what force would keep the particle at rest.

8. Give a geometrical construction, showing a particle in equilibrium under the action of three forces whose magnitudes are 5, 6, and 7 lbs. wt. respectively.

9. Three forces acting on a particle are in equilibrium; their directions are inclined to one another at angles 60°, 150°, and 150°: find the ratios of the forces.

10. A weight of 20 lbs. is suspended freely from a fixed point by a perfectly flexible string: find what horizontal force applied to the body will drag it out so that the string will make an angle of 30° with the vertical, and find the tension of the string.

11. Show that forces of 2, 3, and 6 lbs. wt. could not keep a particle in equilibrium.

12. If $ABCD$ be a quadrilateral, and a particle is acted on by forces parallel and proportional to AB, AD, CB, and CD respectively, prove that the resultant of these forces is represented by four times the line joining the middle points of the diagonals of the quadrilateral.

13. Three forces act on a particle, one of which is given in magnitude and direction: a second is given in direction only. Show that the least force which will keep the particle in equilibrium is perpendicular to the second force.

14. Prove that, if two forces acting at a point O in directions OA and OB be represented in magnitude by $l \,.\, OA$ and $m \,.\, OB$ respectively, their resultant is represented by $(l+m) \,.\, OC$, where C divides AB so that $l : m :: CB : CA$.

15. Assuming that the resultant of two concurrent forces is represented in *direction* by the diagonal of the parallelogram formed by them, deduce from this that the diagonal represents the resultant in *magnitude* also.

16. Find a point inside a triangle such that, if it be acted on by forces represented by lines joining it to the vertices, it will be in equilibrium.

17. Show that forces acting at the angular points of a triangle ABC represented by AL, BM, and CN, where L, M, and N are the middle points of the sides, are in equilibrium.

18. A and B are two fixed points: a particle P is acted on by forces represented by $k \,.\, AP$ and $k \,.\, PB$; show that their resultant is constant in magnitude and direction wherever P may be situated.

CHAPTER III.

RESOLUTION OF FORCES.

31. We have seen in the last chapter that any number of forces acting on a particle (and therefore whose lines of action meet in a point) can be compounded into a single resultant force. The original forces of which this is the resultant are called **components.** Conversely, a single force can be replaced by two or more component forces having that single force for their resultant. This latter process is called **Resolution of Forces.**

In this chapter we shall chiefly be concerned with resolving a single force into *two* component forces *at right angles to one another.*

Resolution in two directions at right angles to one another.

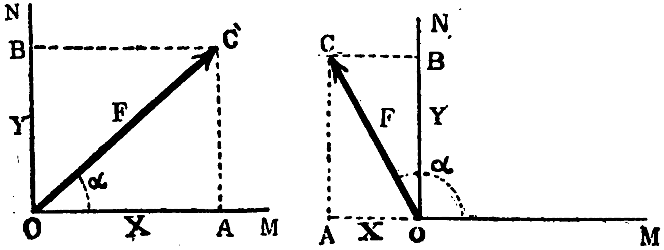

Fig. 13.

32. Let OC represent a force F which it is required to resolve into two components—one along a direction OM, and the other in a direction ON perpendicular to OM.

Through C draw parallels CB and CA parallel to OM and ON respectively; then OA and OB represent the required components, which we will denote by X and Y respectively.

Let the angle COM be called α; then

$$\frac{OA}{OC} = \cos\alpha, \quad \text{or} \quad \frac{X}{F} = \cos\alpha; \quad \therefore X = F\cos\alpha;$$

also,

$$\frac{OB}{OC} = \frac{AC}{OC} = \sin\alpha, \quad \text{or} \quad \frac{Y}{F} = \sin\alpha, \quad \text{or} \quad Y = F\sin\alpha.$$

Hence the required components of F are $F\cos\alpha$ and $F\sin\alpha$ along OM and ON respectively.

If the angle α be obtuse (fig. 13), then the parallel CA cuts MO produced in A. Hence the components of F along AM

$$= -OA = -OC\cos COA = OC\cos\alpha, \quad \text{or} \quad X = F\cos\alpha, \quad \text{as before;}$$

also, the component along ON

$$= OB = OC\sin COA = OC\sin\alpha; \quad \text{or} \quad Y = F\sin\alpha.$$

33. **Definition.**—The **Resolved Part** or **Resolute** of a force along a given direction is the component of the force in that direction *when the other component is perpendicular to it.*

Thus, the resolved part or resolute of F (fig. 13) along the direction OM is X, or $F\cos\alpha$; that is, *the resolved part of a force along any given direction is found by multiplying the force by the cosine of the angle between the direction of the force and the given direction.*

Corollary.—If the direction of the force F is along or parallel to the line OM, its resolved part along OM is F, or, in other words, its full value.

Again, if the direction of F is perpendicular to the line OM, the resolved part of F along $OM = 0$; or, in other words, the resolved part of a force along a direction at right angles to its own line of action is zero.

To illustrate this principle: if we take a key-ring and place it on a smooth pencil or pen-handle, we find that if we pull the ring in the direction of the pencil, we are able to move it with greater ease than if the force were applied in any other direction; whereas, if we pull the ring in a direction at right angles to the pencil, the ring has not the slightest tendency to move along the pencil.

Ex. 1. A force of 10 lbs. wt. acts in a direction inclined at an angle of 60° to the horizontal: find its horizontal and vertical components.

Let its horizontal and vertical components be X and Y respectively (fig. 13).

Here, $$X = 10 \cos 60^\circ = 10 \times \tfrac{1}{2} = 5 \text{ lbs. wt.},$$

and $$Y = 10 \sin 60^\circ = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ lbs. wt.}$$

Ex. 2. A force of 100 lbs. wt. is inclined at an angle of 135° to a given direction. Find its components along and at right angles to that direction.

Let the components along and at right angles to the given direction be X and Y respectively (fig. 13).

Here,

$$X = 100 \cos 135 = 100 \times -\frac{1}{\sqrt{2}} = -\frac{100}{\sqrt{2}} = -\frac{100\sqrt{2}}{2} = -50\sqrt{2} \text{ lbs. wt.}$$

and

$$Y = 100 \sin 135 = 100 \times \frac{1}{\sqrt{2}} = \frac{100}{\sqrt{2}} = \frac{100\sqrt{2}}{2} = 50\sqrt{2} \text{ lbs. wt.}$$

If two forces act on a particle, the resolved part of their resultant in any direction is equal to the algebraic sum of the resolved parts of the forces in the same direction.

34. Let OA, OB, and OC (fig. 14) represent the forces P, Q, and their resultant R respectively; and let OX be any direction whatever. Drop perpendiculars AL, BM, and CN on the line OX. Therefore,

$$\left.\begin{array}{l} OL \text{ represents the resolved part of } P \text{ along } OX \\ OM \text{ represents the resolved part of } Q \text{ along } OX \\ ON \text{ represents the resolved part of } R \text{ along } OX \end{array}\right\} \text{(Art. 33);}$$

and we have to prove that $ON = OL + OM$.

Draw AD parallel to OX.

It is evident (Euclid, I. 26) that the two triangles OBM and ACD are equal in every respect.

$$\therefore\ OM = AD; \text{ but } AD = LN; \quad \therefore\ OM = LN;$$

but $ON = OL + LN$; $\therefore\ ON = OL + OM$. (Q. E. D.)

Corollary.—Since what is true of two forces and their resultant is true of the resultant of that resultant and a third force, and so on, therefore the above theorem is true of any number of forces acting on a particle—that is, whose lines of action meet in a point.

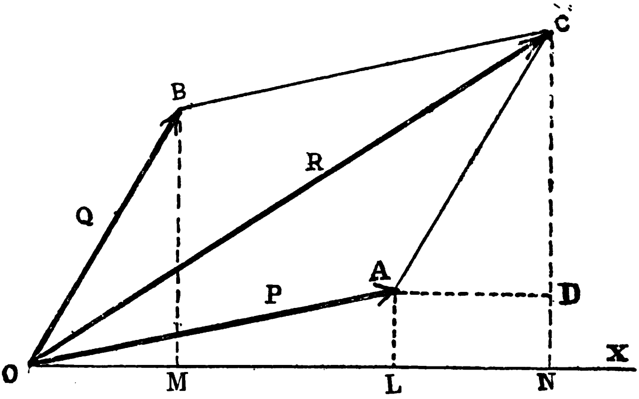

FIG. 14.

To find the Resultant of any number of coplanar Forces acting on a Particle.

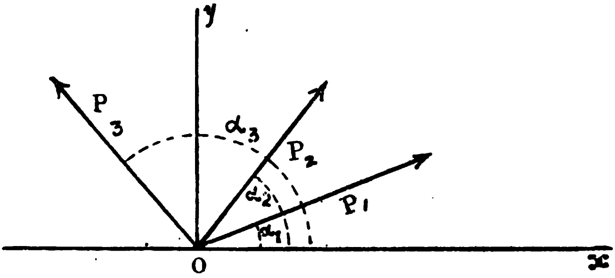

FIG. 15.

35. We have already shown (Art. 29) how to find, by a Graphic construction, the resultant of a number of forces acting on a particle; and we now proceed to find the resultant by actual calculation.

Let the forces be P_1, P_2, P_3, etc. (only three of the forces are represented in fig. 15). Now take any two directions Ox and Oy at right angles to one another, and let a_1, a_2, a_3, etc., be the angles which the directions of P_1, P_2, P_3, etc., respectively make with Ox.

Now P_1 resolves into

$$P_1 \cos a_1 \text{ along } Ox,$$

and

$$P_1 \sin a_1 \text{ along } Oy. \qquad \text{(Art. 33.)}$$

Similarly for P_2, P_3, etc.

Then, if we let the algebraic sum of the components of the forces along Ox and Oy be represented respectively by X and Y, we have

$$X = P_1 \cos a_1 + P_2 \cos a_2 + P_3 \cos a_3 + \ldots$$

and

$$Y = P_1 \sin a_1 + P_2 \sin a_2 + P_3 \sin a_3 + \ldots.$$

The whole system of forces has now been reduced to two forces X and Y at right angles to one another acting at the point O, these being therefore the components of the resultant along Ox and Oy respectively (Art. 34); therefore the resultant

$$R = \sqrt{X^2 + Y^2};$$

and, if θ be the angle which the direction of the resultant makes with Ox, we have

$$\tan \theta = \frac{Y}{X} \text{ (Art. 22)},$$

which gives the direction of the resultant. Hence the following rule for finding the resultant of any number of forces acting on a particle :—

Choose any two directions at right angles to one another, and resolve the forces along each of these directions, denoting the algebraic sum of the results by X and Y respectively ; then the magnitude and direction of the resultant of X and Y will also be the magnitude and direction of the resultant of the original forces.

Ex. 1. The sides *AB* and *BC* of a rectangle *ABCD* are 12′ and 5′ respectively. Forces of 6, 26, and 30 lbs. wt. act along *AB*, *AC*, and *AD* respectively. Find the result in magnitude and direction.

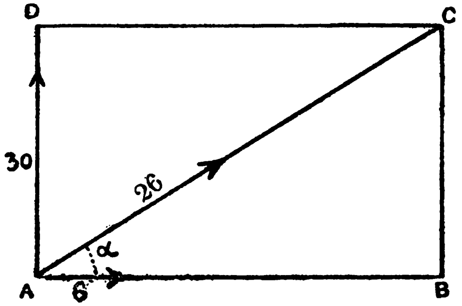

FIG. 16.

Evidently the length of $AC = \sqrt{12^2 + 5^2} = 13$.

The most convenient rectangular directions along which to resolve are evidently *AB* and *AD*.

Now,

The force 6 lbs. has components of 6 along *AB*, and 0 along *AD*,

The force 26 lbs. has components of $26 \cos \alpha$ along *AB*, and $26 \sin \alpha$ along *AD*,

The force 30 lbs. has components of 0 along *AB*, and 30 along *AD*: therefore, if X and Y represent the algebraic sum of the components along *AB* and *AD* respectively, we have

$$X = 6 + 26 \cos \alpha = 6 + 26 \times \frac{12}{13} = 30 \text{ lbs. wt.};$$

and

$$Y = 26 \sin \alpha + 30 = 26 \times \frac{5}{13} + 30 = 40 \text{ lbs. wt.}:$$

therefore the resultant $R = \sqrt{30^2 + 40^2} = 50$ lbs. wt.

Also, if θ be the angle which the direction of R makes with *AB*, we have

$$\tan \theta = \frac{40}{30} = \frac{4}{3}; \quad \therefore \quad \theta = \tan^{-1} \frac{4}{3}.$$

Ex. 2. Four forces, whose magnitudes are equal to $3\sqrt{3}$, 9. 9, and $4\sqrt{3}$ units of weight, act on a particle, the angle between the first and second being 30°, between the second and third 120°, and between the third and fourth 90°. Find the magnitude and direction of their resultant.

As it is always more convenient to make one of the two rectangular directions along which we resolve coincide with the line of action of one of the given forces, therefore take one direction *Ox* along the direction of $3\sqrt{3}$, and the other *Oy* perpendicular to it (fig. 17); and resolving each force into two components along *Ox* and *Oy* respectively, we get the following tabulated results:—

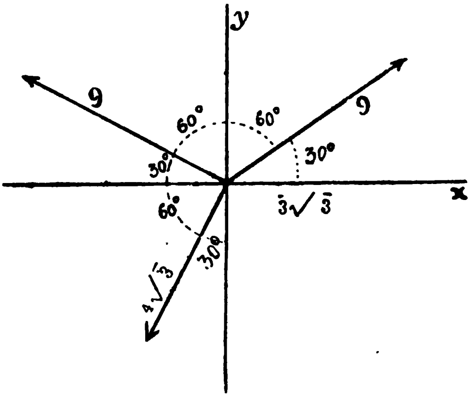

Fig. 17.

Force.	Resolute along *Ox*.	Resolute along *Oy*.
$3\sqrt{3}$	$3\sqrt{3}$	0
9	9 cos 30	9 sin 30
9	− 9 cos 30	9 sin 30
$4\sqrt{3}$	$-4\sqrt{3}\cos 60$	$-4\sqrt{3}\sin 60$

Therefore, adding up the results,

$$X = 3\sqrt{3} + 9\cos 30 - 9\cos 30 - 4\sqrt{3}\cos 60 = \sqrt{3},$$

$$Y = 0 + 9\sin 30 + 9\sin 30 - 4\sqrt{3}\sin 60 = 3;$$

$$\therefore \quad R = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3} \text{ units of weight};$$

and, if θ be the angle which R makes with *Ox*,

$$\tan\theta = \frac{Y}{X} = \frac{3}{\sqrt{3}} = \sqrt{3}; \quad \therefore \quad \theta = 60°.$$

Ex. 3. Show how a vessel can sail partly against the wind.

Let AOB be the direction of the relative velocity of the wind, and COD the direction of the ship's length: also, let OM be the direction of the sail which must lie within the angle BOC. Let the angle $BOM = \alpha$.

Now, any force F acting along OB resolves itself into two components, $F \cos \alpha$ along the sail OM, and $F \sin \alpha$ perpendicular to OM. But $F \cos \alpha$ along the sail may be rejected; and therefore we need only consider $F \sin \alpha$ perpendicular to the sail. If θ be the angle which this force makes with OD, therefore $F \sin \alpha$ resolves itself into $F \sin \alpha \cos \theta$ along OD, the direction of the length of the boat, and $F \sin \alpha \sin \theta$ perpendicular to this length.

But the latter component being perpendicular to the direction of the keel is almost neutralized by the reaction of the water. Therefore, there is only one force left which acts along OD, and the path of the ship will be nearly along this line.

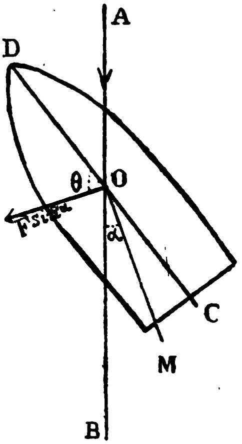

FIG. 18.

36. To find the conditions that a number of forces acting on a particle should be in equilibrium.

Let X and Y be the algebraic sums of the resolved parts of the forces along two rectangular directions Ox and Oy respectively (fig. 15); then (Art. 35), $R^2 = X^2 + Y^2$. But if the forces are in equilibrium, the resultant vanishes; hence $X^2 + Y^2 = 0$.

But, since the square of a real quantity must be positive, it follows that the sum of the squares of real quantities could not equal zero unless each is separately zero.

$$\therefore \quad X = 0, \quad \text{and} \quad Y = 0.$$

Hence, *if a number of forces acting on a particle be in equilibrium, the algebraic sums of their resolved parts in two directions at right angles to one another are separately zero.*

Conversely, *if the sums of the resolved parts in two directions at right angles to one another are separately zero, the forces are in equilibrium.*

For, in this case, each of the components X and Y is zero, hence, R is zero, and therefore the forces are in equilibrium.

EXAMPLES IV.

1. A force of 20 lbs. wt. acts in a direction inclined at an angle of 45° to the horizontal: find its horizontal and vertical components.

2. If the resolved part of a force in a given direction to which it is inclined at an angle of 60° be $20\sqrt{3}$ lbs. wt., find the magnitude of the force.

3. A canal-boat in the centre of a canal 10 yards broad is being towed by a horse on the bank by means of a rope 13 yards long: if the horse pulls with a force of 260 lbs. wt., find the force with which the boat is pulled along the direction of the canal.

4. Why is a very long rope used in drawing a canal-boat, and not a short one?

5. Find the resultant of forces of 32, 20, and 8 lbs. wt. acting at A in the directions AB, AC, and AD respectively, where $ABCD$ is a rectangle whose side $AB = 4$ inches, and $BC = 3$ inches.

6. A weight of 40 lbs. is supported by three strings; one is pulled horizontallv with a force of 9 lbs. wt., the second is vertical, and the third inclined at

$$\sin^{-1}\frac{3}{5}$$

to the vertical: find the tensions of the second and third strings.

7. Forces equal to 2, 3, 4, and 5 lbs. wt. act on a particle in directions north, south, east, and west respectively; also forces 6 and 8 lbs. wt. act in directions south-east and south-west respectively: find the resultant of all the forces.

8. A particle of mass 24 lbs. is suspended from a fixed point by a string 13″ long. Find what horizontal force applied to the body will draw it out until it is at a perpendicular distance of 5″ from the vertical drawn through the point of suspension. Also find the tension of the string.

CHAPTER IV.

MOMENTS.

37. If a body in the form of a flat sheet or lamina, such as a sheet of paper or cardboard, be taken and laid on a table, then if any point *O* (fig. 19) be fixed by means of a pin driven into the table, and a force *P* applied, it is found that the effect of the force is to cause the body to rotate round the point *O*. It is also found that the farther the line of action of the force is from *O* the greater is the effect in rotating the body.

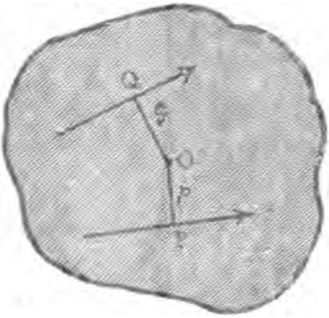

Fig. 19.

Again, if a force whose direction passes through the pin at *O* be applied, no rotatory movement is produced in the body.

The same result is experienced in the case of a body which can move freely round a fixed axis, such as a door on its hinges. If we apply a force in a direction perpendicular to the plane of the door at a point of the door furthest from the hinge, it is found that a very small effort is sufficient to cause the door to rotate. On the other hand, a very much greater force would be necessary to rotate the door with the same angular velocity as before if the force were applied close to the hinge; and if the line of action of the force pass through any point of the hinge, the door will not rotate at all.

38. We thus see that the tendency of a force to rotate a body about a fixed point depends on two things:—

(1) *The magnitude of the force.*

(2) *The perpendicular distance of the point from the line of action of the force.*

Hence the following definition :—

Definition.—*The* **moment** *of a force about a fixed point is the product of the force and the perpendicular distance of the point from its line of action.*

The physical idea of a moment, as we have seen above, is a rotation or twist.

39. **Positive and negative Moments.**—A moment may represent either a rotation in the same direction as that in which the hands of a clock or watch revolve, or the reverse. It is really immaterial which we regard as positive and which negative. A contra-clockwise rotation, however, is generally considered positive and a clockwise rotation negative. For example, if P and Q be two forces acting on a body (fig. 19) whose perpendicular distances from a point O on the body are p and q respectively ; then

The moment of P round $O = +Pp$ (being contra-clockwise), and

The moment of Q round $O = -Qq$ (being clockwise).

Geometrical Representation of the Moment of a Force about a Point.

40. Let a force P, represented by the line AB (fig. 20), act on a body, and let p = the length of the perpendicular from any point O in the body on the line of action of P.

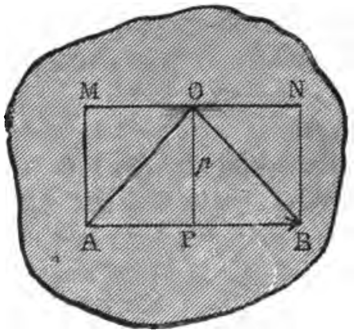

Fig. 20.

Join OA and OB; and on AB construct a rectangle $AMNB$ of height $= p$.

$\therefore$ moment of P about $O = P \times p$; i.e. $AB \times p$; but

$AB \times p$ = area of the rectangle $AMNB = 2 \triangle AOB$.

Hence the moment of a force about a point is represented by *twice the area of the triangle whose base is the line representing the force, and whose vertex is at the point.*

41. **Varignon's Theorem of Moments.**—*The algebraic sum of the moments of any two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point.*

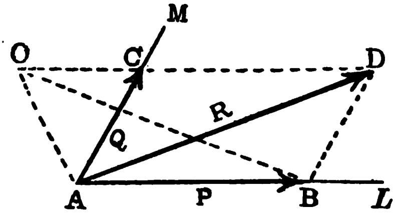

Fig. 21 (*a*)

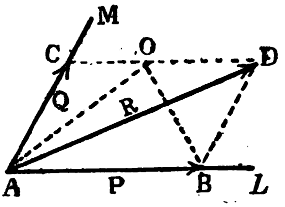

Fig. 21 (*b*).

Let the forces P and Q act along the lines AL and AM, respectively, and let O be any point in their plane (fig. 21 (a) and (b)). Through O draw a parallel to AL, meeting AM in C. Then choose the units of length which represent units of force on such a scale that AC represents the force Q;* and on the same scale cut off AB to represent P; complete the parallelogram $ABDC$, and AD will represent the resultant R.

42. **Case I.**—Where O lies outside the angle CAB (fig. 21 (a)).

Join OA and OB.

Now the moment of P round O is represented by $2 \triangle AOB$ (Art. 40); but $2 \triangle AOB = 2 \triangle ADB$ (Euclid, I. 37) $= 2 \triangle ACD$ (I. 34), and the moment of Q round O is represented by $2 \triangle AOC$; therefore the sum of the moments of P and $Q = 2 \triangle ACD + 2 \triangle AOC = 2 \triangle AOD$; but $2 \triangle AOD$ represents the moment of R round O.

43. **Case II.**—Where O lies inside the angle CAB (fig. 21 (b)).

The moment of P round O is represented by $2 \triangle AOB$; but $2 \triangle AOB = 2 \triangle ADB$ (I. 37) $= 2 \triangle ACD$ (I. 34), and the moment of Q round O is represented by $-2 \triangle AOC$ (clockwise); therefore the algebraic sum of the moments of P and Q $= 2 \triangle ACD - 2 \triangle AOC = 2 \triangle AOD$; but $2 \triangle AOD$ represents the moment of R round O, as before. Hence, generally, the moment of P + moment of Q = moment of R.

NOTE.—It is to be borne in mind that, in speaking of the *algebraic sum* of moments, we mean that the plus or minus signs of the moments have to be considered, and that if the signs of the two moments which are to be added are unlike, then the term *algebraic sum* really means the difference of the actual quantities which measure the moments.

44. A particular case of this theorem is when the point O is situated on the line of action of the resultant R; then the moment of R round O vanishes, and therefore the algebraic sum of the moments of P and Q round O vanishes;

---

* The student will see that in this proof the units must be chosen so that the point O may lie on one side of the parallelogram AD. Another proof, which is unrestricted by this condition, is given in the Appendix.

and hence, *the moments of two intersecting forces about any point on the line of action of their resultant are equal and opposite.*

Conversely, *if the moments of two intersecting forces about any point in their plane be equal and opposite, then either their resultant is zero or the point is situated on the line of action of their resultant.*

These propositions can easily be proved independently of the previous proof; for, in the former case, O coincides with D, and the moments of P and Q round O are measured by $2 \triangle ABD$ and $-2 \triangle ACD$ respectively, and are, therefore, equal and opposite. The converse is also evident.

45. In the next chapter it will be shown that Varignon's theorem of moments is true, not only for two forces whose lines of action intersect, but also for two like or unlike parallel forces, the only exception being, as will be seen in the next chapter, when the two parallel forces are equal and unlike, the pair constituting what is called a **couple**, in which case there cannot be any resultant.

46. *If any number of coplanar forces have a resultant, the algebraic sum of the moments of the forces about any point in their plane is equal to the moment of their resultant about that point.*

For, the above theorem of moments being true for any two forces and their resultant, must also be true of the resultant of this resultant and a third force, and must therefore be true of any three forces and their resultant; similarly, it is true for four or for any greater number of forces.

47. From this it follows, as in the case of two forces, that *the algebraic sum of the moments of any number of coplanar forces about a point situated on the line of action of their resultant is zero,* and conversely, *if the algebraic sum of the moments of any number of coplanar forces about a point in their plane be zero, then either their resultant is zero or the line of action of their resultant passes through the point.*

Ex. **Forces act along the sides AB, AD, CB, and CD of a square $ABCD$ equal respectively to 10, 20, 30, and 40 lbs. wt.; also a force of $16\sqrt{2}$ lbs. wt. acts along BD: find the algebraic sum of the moments of these forces about the centre of the square, and also about the point A, the side of the square being 6 feet long.**

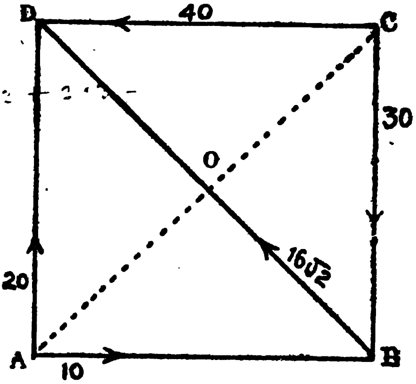

FIG. 22.

The moments of the forces 40 and 10 about the centre O being *contra-clockwise* are *positive*, while those of the forces 30 and 20 being *clockwise* are *negative*; while the moment of $16\sqrt{2}$ about O is zero.

Also the perpendicular distance of O from each of the forces 10, 20, 30, and 40 is 3 feet.

Hence the algebraic sum of moments about O is

$$10 \times 3 - 20 \times 3 - 30 \times 3 + 40 \times 3 = 0.$$

Again, the moment of each of the forces 10 and 20 about A is zero; also the moment of the force $16\sqrt{2}$ about A is

$$16\sqrt{2} \times AO = 16\sqrt{2} \times 3\sqrt{2}$$

and is positive.

Hence the algebraic sum of moments about A is

$$\begin{aligned} 40 \times 6 - 30 \times 6 + 16\sqrt{2} \times 3\sqrt{2} &= 40 \times 6 - 30 \times 6 + 96 \\ &= 156 \text{ units of moment} \end{aligned}$$

(taking as the unit of moment the moment of a force of 1 lb. wt. round a point whose perpendicular distance from the line of action of the force is one foot).

EXAMPLES V.

1. Along the sides AB, CB, DC, and DA of a rectangle $ABCD$ forces act equal to 6, 12, 18, and 4 lbs. weight respectively. If the side AB be 10 inches long and B 8 inches, find the algebraic sum of the moment of the forces (1) about the point of intersection of the diagonals, (2) about the point A.

2. Three forces of 20, 30, and 40 lbs. wt. act along the sides BC, CA, and AB respectively of a triangle. If the lengths of B, C, A, and AB be 13, 5, and 12 feet respectively, find the algebraic sum of the moments of the forces round each of the three vertices of the triangle.

3. A uniform beam AB, 10 feet long, weighing 80 lbs., can turn about a smooth hinge at A: what horizontal force, acting at B, will enable the beam to rest with its centre 4 feet below the level of A?

Here the beam being uniform, its weight may be supposed to act at its middle point. Then, if P be the required horizontal force, the beam is in equilibrium under the action of P, the weight 80 lbs., and the reaction of the hinge. Hence, if we take moments round the hinge A we have, remembering that the reaction of the hinge has no moment round A,

$$P \times 8 - 80 \times 3 = 0; \quad \therefore\ P = 30 \text{ lbs. wt.}$$

4. A rectangular trap-door $ABCD$ has a hinge along the side AB which is horizontal. Its weight is 60 lbs.: find what vertical force must be applied at the middle point of the edge CD in order to keep the door in a horizontal position, assuming the weight of the door to act at its middle point, given

$$BC = AD = 6 \text{ feet.}$$

5. In the last example, if the door be kept in a horizontal position by means of a rope applied at the middle point of CD, and whose direction is perpendicular to CD and makes an angle of 30° with the horizon, find the tension of the rope.

6. A carriage wheel, whose weight is W, and radius r, rests upon a level road: show that the horizontal force P necessary to draw the wheel over an obstacle of height h is given by the equation

$$P = W\frac{\sqrt{h(2r-h)}}{r-h}.$$

N.B.—Take moments round the summit of the obstacle.

7. A pole AB, 20 feet long, is in a vertical position with the end B resting on the ground. A horizontal wire attached to the extremity A has a tension of 40 lbs. weight. The pole is kept in a vertical position by a wire-stay fixed to the pole at a point 8 feet above B, and into the ground at a point 6 feet from B. Find the tension of the wire-stay.

8. Forces act along the sides of a triangle taken in order, each force being proportional to the side along which it acts. Show that the moments of the forces about the opposite vertices are all equal, each being represented by twice the area of the triangle.

9. In example 8 show that the algebraic sum of the moments of the forces about any point in their plane is a constant quantity which is represented by twice the area of the triangle.

10. The two portions of a horizontal wire passing round the top of a vertical telegraph pole are inclined at an angle of 60° to one another. The pole is supported by a wire-stay attached to the middle point of the pole, and inclined at 60° to the horizon. If the tension of each portion of the telegraph wire be 100 lbs. wt., find the tension of the stay.

11. A vertical pole, 30 feet long, rests with one end on the ground, and is pulled by a string attached to its upper end inclined at 30° to the horizon. If the tension of the string be 100 lbs. wt., find what horizontal force applied at a point 6 feet above the ground will keep the pole in a vertical position.

12. Parallel forces of 11, 12, and 13 lbs. wt. are such that their lines of action are at distances 4, 5, and 6 feet respectively from a fixed point. If the first of these forces be in opposite direction to the other two, find the algebraic sum of their moments about the point.

CHAPTER V.

PARALLEL FORCES.

48. Having shown how two or more concurrent forces (i.e. forces meeting in a point) can be compounded into one single resultant force, we proceed to the consideration of parallel forces.

Definition.—Parallel forces are said to be "**like,**" when they act in the same direction; and "**unlike,**" when they act in opposite directions.

To find the Resultant of two Parallel Forces acting on a Rigid Body.

Case I.—To find the resultant of two **like** parallel forces.

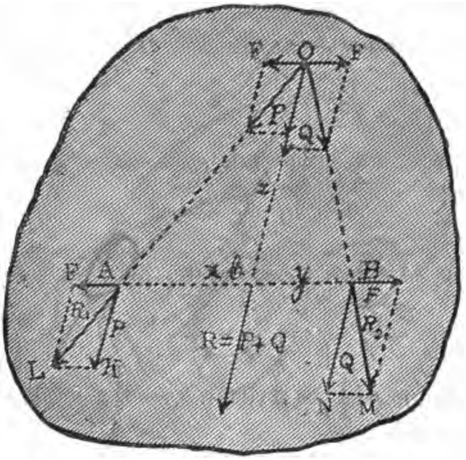

FIG. 23.

Let P and Q be the *like* parallel forces acting at the points A and B respectively.

At A and B apply two equal and opposite forces, each equal to F. These forces will not disturb the system (axiom 2, Art. 18).

Compound P and F at A into a resultant R_1, and Q and F at B into a resultant R_2.

Produce the lines of action of R_1 and R_2 to meet at O, and transfer the points of application of R_1 and R_2 to the point O (axiom 1, Art. 17).

Then at O resolve R_1 into its original components P and F, and R_2 into its original components Q and F.

Now, the two forces at O, each equal to F, being equal and opposite, neutralize one another, and may therefore be removed.

There now only remain two forces, P and Q, acting at O in the same direction, whose resultant R is therefore equal to their sum; therefore, $R = P + Q$, and whose point of application may be transferred to C (axiom 1, Art. 17).

Again, let the lengths of the segments AC and BC be denoted by x and y respectively, and the length of OC by z.

Since the triangles OCA and AKL are equiangular, their sides are proportional (Euclid, VI. 4). Hence,

$$\frac{AK}{KL} = \frac{OC}{CA}, \quad \text{or} \quad \frac{P}{F} = \frac{z}{x}.$$

And similarly, the two triangles OCB and BNM are equiangular;

$$\therefore \quad \frac{Q}{F} = \frac{z}{y}.$$

Dividing the former of these results by the latter, we obtain

$$\frac{P}{Q} = \frac{y}{x}.$$

This result might also be written $Px = Qy$. We thus have the following rule:—

The resultant of two like parallel forces is in the same direction as the forces: its magnitude is equal to the sum of the forces, and its point of application divides the line joining the points of application of the forces inversely as the forces.

Corollary.—From the above, it is evident that the position of the point *C*, and also the magnitude of the resultant, depend solely on the magnitudes of *P* and *Q*, and not on their directions; and therefore, if a pair of like parallel forces have their lines of action turned through a common angle, and hence still remain parallel to one another, the point at which the resultant acts is not altered, and the magnitude of the resultant is still equal to the sum of the forces.

49. **Case II.**—To find the resultant of two **unlike** parallel forces.

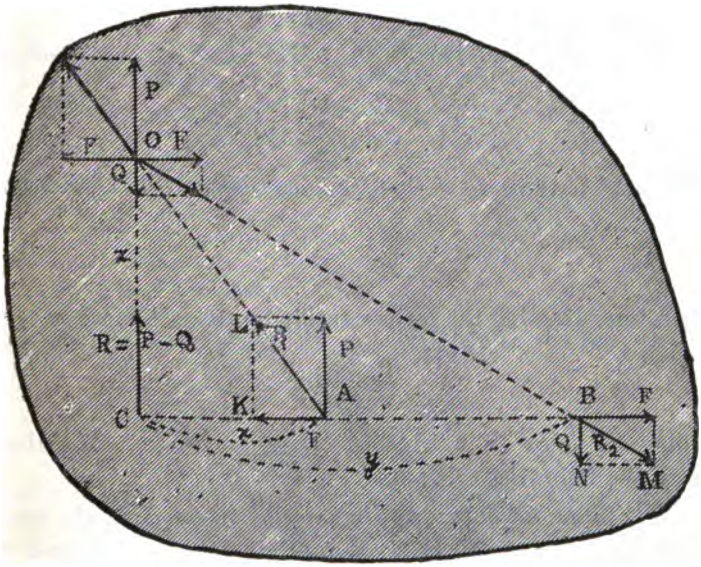

FIG. 24.

Let *P* and *Q* be two unlike parallel forces acting at the points *A* and *B* respectively, *P* being the greater force.

At *A* and *B* apply two equal and opposite forces, each equal to *F*, as before.

Compound *P* and *F* at *A* into a resultant R_1, and *Q* and *F* at *B* into a resultant R_2.

Produce the lines of action of R_1 and R_2 to meet at O.

Transfer R_1 and R_2 so as to act at O.

At O resolve R_1 into its original components P and F, and R_2 into its original components Q and F.

The two forces at O, each equal to F, being equal and opposite, may be removed.

There now only remain two forces P and Q acting at O in opposite directions whose resultant is equal to their difference; therefore, $R = P - Q$, and is parallel to the directions of P and Q; and the point of application of R may, as before, be transferred to the point C on BA produced.

Again, let the lengths AC and BC be denoted by x and y respectively, and the length of OC by z.

Now, since the triangles OCA and LKA are equiangular,

$$\therefore \frac{LK}{KA} = \frac{OC}{CA}, \quad \text{or} \quad \frac{P}{F} = \frac{z}{x}.$$

Similarly, in the case of the triangles OCB and BNM, we have

$$\frac{Q}{F} = \frac{z}{y};$$

therefore, dividing the former result by the latter, we have

$$\frac{P}{Q} = \frac{y}{x}.$$

This may also be written $Px = Qy$, as before. We thus have the following rule:—

The resultant of two unlike parallel forces is in the same direction as the greater of the forces; its magnitude is equal to the difference of the forces, and its point of application divides the line joining the points of application of the forces externally inversely as the forces.

It follows also, as in the case of like parallel forces, that the magnitude of R and the position of C depend solely on the magnitudes of P and Q, and not on their directions, and are therefore unaltered when the lines of action of the forces are turned through a common angle.

NOTE 1.—It is evident that in all cases, whether the forces be like or unlike, the point C at which the resultant acts is closer to the greater of the two forces.

NOTE 2.—Generally, the resultant of any two parallel forces, whether like or unlike, is equal to their *algebraic sum*—that is, the sum of the two forces, giving each its proper sign.

50. *To prove that the resultant of any number of parallel forces is equal to their algebraic sum.*

For if P, Q, R, etc., be the parallel forces, then the resultant of P and Q is their sum $P + Q$ (each being taken with its proper sign), and is parallel to each of the forces. Also the resultant of $P + Q$ and R is their algebraic sum $P + Q + R$, and is parallel to each of the forces; the same is true of this resultant and a fourth force; hence it is true of any number of parallel forces.

Ex. 1. Find the resultant of two like parallel forces of 3 lbs. and 5 lbs. wt. respectively, acting at points 14 inches apart.

Here, $R = 3 + 5 = 8$ lbs. wt.

Also, let $AC = x$ (fig. 23); $\therefore$ $BC = 14 - x$,

and let the forces of 3 lbs. and 5 lbs. act at A and B, respectively;

$$\therefore \quad \frac{3}{5} = \frac{BC}{AC} = \frac{14 - x}{x};$$

or

$$3x = 5(14 - x) \quad \therefore \quad x = 8\tfrac{3}{4}''.$$

Hence the resultant = 8 lbs. wt., and acts at a point $8\frac{3}{4}$ inches from the point of application of the 3 lbs. force.

Ex. 2. The resultant of two unlike parallel forces is a force of 10 lbs. wt., and acts at a point 16 inches from the point of application of the greater force. The points of application of the forces are 20 inches apart: find the magnitudes of the forces.

Here (fig. 24) $AC = 16$ inches and $BC = 36$ inches.

Hence,

$$\frac{P}{Q} = \frac{36}{16} = \frac{9}{4}, \quad \text{or} \quad 4P - 9Q = 0;$$

also,

$$P - Q = 10.$$

And, solving, we obtain $P = 18$ lbs. wt., and $Q = 8$ lbs. wt.

51. To prove that *the algebraic sum of the moments of two parallel forces round any point in their plane is equal to the moment of their resultant about the same point* (Varignon's Theorem of Moments).

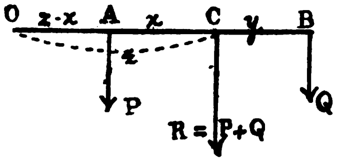

Fig. 25 (*a*).

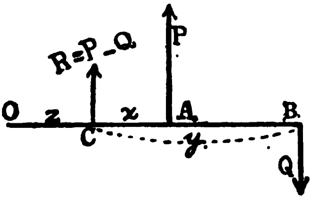

Fig. 25 (*b*).

Case I.—Where the forces are **like.**

Let P and Q be the two forces (fig. 25 (a)), and R or $P + Q$ their resultant. Then, if O be any point in their plane, through O draw a line perpendicular to P, Q, and R, and cutting them in A, B, and C respectively; and we may (axiom 1, Art. 17) regard A, B, and C as the respective points of application of these forces.

Hence, we have $Px = Qy$ (Art. 48).

Also, if $OC = z$,

Therefore the sum of the moments of P and Q about O

$$= P \,.\, OA + Q \,.\, OB = P(z - x) + Q(z + y)$$
$$= (P + Q)z + Qy - Px = (P + Q)z \text{ (since } Px = Qy)$$
$$= Rz = \text{moment of } R \text{ about } O.$$

Case II.—Where the forces are **unlike.**

In this case $R = P - Q$, and $Px = Qy$ (Art. 49).

Here, if $OC = z$, as before (fig. 25 (b)), the algebraic sum of the moments of P and Q about O

$$= P \,.\, OA - Q \,.\, OB = P(z + x) - Q(z + y)$$
$$= (P - Q)z + Px - Qy = (P - Q)z \text{ (since } Px = Qy)$$
$$= Rz = \text{moment of } R \text{ about } O.$$

Note.—It is left as an exercise for the student to prove the truth of the above proposition when the point O is situated between the points A and B.

Corollary.—It follows from the above proposition (as in Art. 46) that the moment of the resultant of *any number* of parallel forces about any point in their plane is equal to the algebraic sum of the moments of the forces about the same point.

Ex. 1. A see-saw plank, 15 feet long, weighs 1 cwt.: find where the support should be placed when two children, weighing 6 and 7 stone respectively, are at the ends, the plank being uniform.

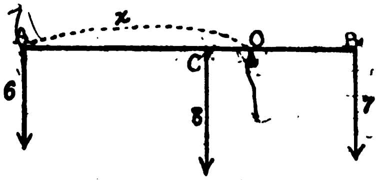

FIG. 26.

Here, the plank AB being uniform, the weight of the plank, viz. 8 ~~stone~~, may be supposed to act at its middle point C. There are therefore three parallel forces of 6, 8, and 7 stones acting at A, C, and B respectively.

The magnitude of the resultant = 6 + 8 + 7 = 21.

Let O be the point of the plank where this resultant acts, and let $AO = x$.

Then, taking moments round the point A, we have

moment of resultant = sum of moments of forces:

$$\therefore\quad 21x = 6 \times 0 + 8 \times 7\tfrac{1}{2} + 7 \times 15,$$

$$21x = 60 + 105 = 165\,;$$

$$\therefore\quad x = 7\tfrac{6}{7}\text{ feet}.$$

Hence, the point of support is $7\frac{6}{7}$ feet from A.

Ex. 2. A uniform beam, 20 feet long, weighing 80 lbs., is supported horizontally by props at distances of 4 and 18 feet from one end, and carries weights of 15, 12, 13, and 8 lbs. at distances of 0, 7, 14, and 20 feet respectively from the same end: find the pressures on the props.

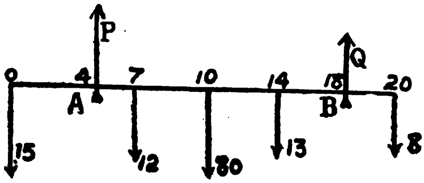

FIG. 27.

Let P and Q be the upward reactions of the props.

Here, the beam is in equilibrium under the action of P and Q upwards and the weights downwards; the resultant of the forces must therefore be zero.

But the moment of resultant = algebraic sum of moments of the forces.

Hence, the algebraic sum of the moments of the forces about *any point* in their plane is zero. Therefore, taking moments about the prop A (the student may note that one of the props is chosen as the point round which moments are taken, as the reaction P at that point has no moment round A, and therefore vanishes from the equation),

$$15 \times 4 - 12 \times 3 - 80 \times 6 - 13 \times 10 + Q \times 14 - 8 \times 16 = 0;$$
$$\therefore \quad Q = 51 \text{ lbs. wt.}$$

Similarly, P might be found by taking moments round the prop B; but it can be found more easily by equating the sum of P and Q to the sum of all the downward forces (the forces being in equilibrium);

$$\therefore \quad P + Q = 15 + 12 + 80 + 13 + 8 = 128, \quad P + 51 = 128;$$
$$\therefore \quad P = 77 \text{ lbs. wt.}$$

EXAMPLES VI.

1. A uniform beam, 3 feet long, the weight of which is 10 lbs., is supported in a horizontal position across a rail with weights of 4 and 13 lbs. suspended from the ends: find the point of the beam in contact with the rail.

2. At what point can a rod, 8 feet long, and weighing 1 lb. per foot, and having weights of 3 and 5 lbs. fastened to the ends, be supported?

3. Weights of 5 lbs. and 19 lbs. are attached at the extremities of a uniform rod of length 10 feet: if the system balance round a fulcrum which is 3 feet distant from one end, determine the weight of the rod.

4. A uniform rod, whose length is 15 inches, and weight 2 lbs., is placed horizontally on a vertical prop distant 3 inches from one extremity at which a weight of 10 lbs. is suspended: what weight must be hung from the other end in order that there should be equilibrium?

5. A uniform beam, 20 feet long, supported horizontally at its extremities, and weighing 5 tons, supports a weight of 3 tons at a distance of 6 feet from one end; calculate the pressures on the supports.

6. A beam, 12 feet long, weighing 60 lbs., supported horizontally by props at its extremities, has a weight of 48 lbs. suspended at a point 3 feet from one end, and an upward thrust of 24 lbs. wt. applied at a point 3 feet from the other end: find the pressures on the supports.

7. A horizontal bar is acted on by vertically upward forces of 5, 6, 4, and 7 lbs. wt., at points distant 0, 5, 7, and 14 inches respectively from a fixed point on the bar, and by vertically downward forces of 8 and 10 lbs. wt. at distances of 3 and 10 inches respectively from the same fixed point: find the magnitude and position of the resultant, and the sum of the moments of all the forces about the points where the 8 lbs wt. and the 10 lbs. wt. act.

8. A uniform beam is 12 feet long and weighs 100 lbs.; if it is supported by props at its extremities, where must a weight of 110 lbs. be hung so that the pressure on one prop may be twice that on the other?

9. On a light bar forces of 7, 10, and 8 lbs. wt. act in the same direction perpendicular to the bar at distances of 0, 10, and 20 feet respectively from one end, and forces of 12 and 9 lbs. wt. act in the opposite direction at distances 4 and 16 feet respectively from the same end: in what two positions may a parallel force of magnitude 8 lbs. wt. act, so that the resultant of all the forces may be at a point distant 13 feet from the same end?

10. A uniform beam, 12 feet long, and weighing 1 cwt., rests on a horizontal table with one end projecting 1 foot over the edge: find the greatest weight which can be hung on the end without making the beam topple over.

11. A uniform plank, of weight 20 stone, and length 18 feet, rests on a horizontal table with 7 feet projecting over the edge of the table; if a weight of 4 stone be placed on the end which rests on the table, how far can a man whose weight is 12 stone walk along the plank from the edge of the table before the plank overturns?

12. Assuming the rule for finding the magnitude and position of the resultant of two like parallel forces, deduce that for finding the resultant of two unlike parallel forces in magnitude and position.

13. $ABCD$ is a square. Forces of 3, 4, and 5 lbs. wt. act along AB, BC, and CD respectively: find the magnitude and direction of their resultant, and the distance of its line of action from C.

14. Forces of 4, 7, 8, and 10 lbs. wt. act along the sides of a square $ABCD$ in directions AB, BC, CD, and DA respectively: find the magnitude and direction of the resultant, and the distance of its line of action from A.

Couples.

52. The method of finding the resultant of two unlike parallel forces (Art. 49) fails when the forces are equal; for (fig. 24) the lines of action of the two resultant forces, R_1 and R_2, would themselves be parallel to one another, and would therefore never meet.

Definition.—Two equal and unlike parallel forces acting at different points of a rigid body are called a **couple**.

Definition.—The **arm** of a couple is the perpendicular distance between the lines of action of the two forces.

Definition.—The **moment** of a couple is the product of either force and the arm.

Thus the two parallel forces P and P, acting at the points A and B of a rigid body, constitute a couple; the perpendicular distance p between the lines of action of the forces is the *arm*, and the moment of the couple = Pp.

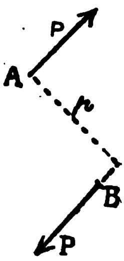

FIG. 28.

53. It is evident that when a couple acts on a body which is free to move it causes the body to rotate. Couples which rotate a body in the same direction are said to be **like**; and if in opposite directions, they are **unlike**.

Examples of Couples.—Many instances of couples will readily occur to the student; for example, when winding a watch, we apply a couple with the finger and thumb to the key, thus causing it to rotate; also, in steering a bicycle so as to cause it to turn to the right or left, if we pull one extremity of the handle with the same force as we push the other end in a parallel and opposite direction, the pair of forces thus applied constitutes a couple.

54. **The algebraic sum of the moments of the two forces of a couple about any point in their plane is constant and equal to the moment of the couple.**

Let two forces each equal to P constitute the couple, and let O be any point in their plane.

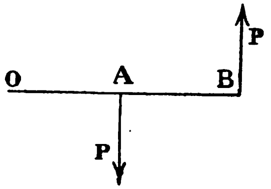

FIG. 29 (*a*).

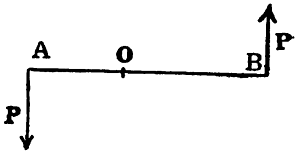

FIG. 29 (*b*).

Through O draw OAB perpendicular to the line of action of the forces, and the forces may be supposed to act at A and B (axiom 1, Art. 17).

(1) If O does not lie between A and B (fig. 29*a*), we have the algebraic sum of moments about O

$$= P.OB - P.OA = P(OB - OA) = P.AB.$$

(2) If O lies between A and B (fig. 29*b*), we have the algebraic sum of moments about O

$$= P.OB + P.OA = P(OB + OA) = P.AB,$$

which in each case is the moment of the couple.

Hence, the algebraic sum of the moments of the forces about O is *independent of the position of* O, being equal to a constant quantity, viz. the moment of the couple. Therefore the **moment** of a couple might also have been defined as *the algebraic sum of the moments of the forces constituting the couple about any point in their plane.*

55. **Two coplanar couples whose moments are equal and opposite will balance one another.**

Let one couple consist of two forces (P, P), having an arm of length p; and let the other couple consist of two forces (Q, Q), having an arm of length q.

Case I.—Where the forces (P, P) of one couple are not parallel to the forces (Q, Q) of the other couple.

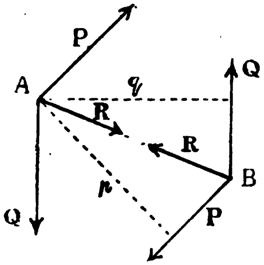

FIG. 30.

Produce the lines of action of the forces of each couple out, and let the line of action of one of the forces P meet that of one of the forces Q in the point A (fig. 30); and let the lines of action of the other two forces meet in B. Hence the forces may be transferred to A and B respectively (axiom 1).

Now, since the moments of the two couples are equal, we have $P.p = Q.q$, where p and q are the arms of the couples. Hence the moments of the two forces P and Q acting at B round the point A are equal and opposite; therefore (Art. 44) the resultant R acting at B must pass through A. Similarly the resultant R (the two resultants being evidently equal) of P and Q acting at A must pass through B. Hence the two resultants, being equal and opposite, neutralize one another, and therefore the two couples are in equilibrium.

56. **Case II.**—Where the forces constituting the couples are all parallel.

Let a straight line perpendicular to the lines of action of the forces meet them in the points A, B, C, D (fig. 31).

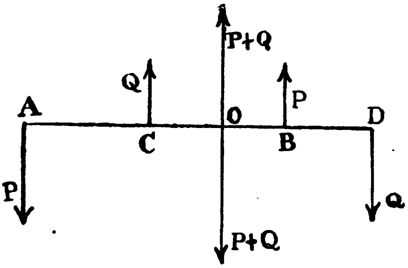

FIG. 31.

Now, if O be the point of application of the resultant $(P + Q)$ of the like parallel forces P at A, and Q at D, we have (Art. 48)

$$P.AO = Q.DO,$$

but we are given $P.AB = Q.CD$;

therefore, subtracting the upper equation from the lower, we get

$$P(AB - AO) = Q(CD - DO),$$

or $$P.BO = Q.CO.$$

Hence O is also the point of application of the resultant $(P + Q)$ of the like parallel forces P at B and Q at C; hence the two resultants, being equal and opposite, neutralize one another.

Corollary.—Since two couples of equal and opposite moments equilibrate, it follows that if the moment of one of the couples be reversed, the two couples will be equivalent to one another; and hence *a couple may be shifted anywhere in its own plane without altering its statical effect.*

57. **Composition of Couples.—The resultant of any number of coplanar couples is a single couple whose moment is equal to the algebraic sum of the moments of the given couples.**

Supposing we have three couples, viz.:—

a couple (P, P) with arm p whose moment $= P.p$,
a couple (Q, Q) with arm q whose moment $= Q.q$,
a couple (R, R) with arm r whose moment $= R.r$,

and let each of them be changed into an equivalent couple, the arm of each being any chosen straight line of length x (fig. 32).

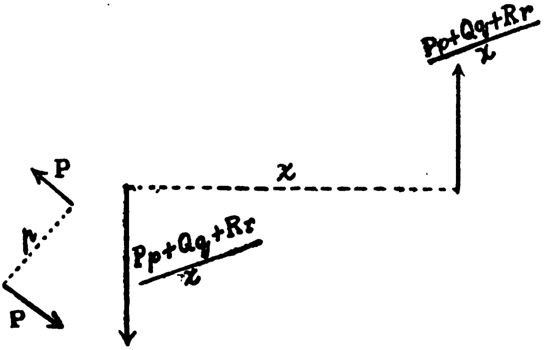

Fig. 32.

Now the couple (P, P) is evidently equivalent to a couple (*Cor.* Art. 56), each of whose forces is

$$\frac{Pp}{x}$$

and whose arm is x, and, therefore, whose moment is still $P.p$.

Similarly for the other two couples.

Hence the three couples are equivalent to a single couple, each of whose forces is

$$\frac{Pp}{x} + \frac{Qq}{x} + \frac{Rr}{x} \text{ or } \frac{Pp + Qq + Rr}{x},$$

the length of the arm being x, and whose moment is

$$Pp + Qq + Rr,$$

due account being taken of the proper signs.

It is evident that this proof will apply to any number of couples.

58. **A force acting at any point A of a rigid body can be replaced by an equal force acting in the same direction at any other point B, together with a couple whose moment is the moment of the original force about B.**

Let P be the original force acting at A, and let B be any other point.

Introduce two equal and opposite forces P and $-P$ acting at B each equal and parallel to the force P at A.

These forces neutralize one another (axiom 2).

But P at A and $-P$ at B constitute a couple whose moment is $P.p$, where p is the length of the arm.

Also $P.p$ is the moment of the original force P acting at A round the point B.

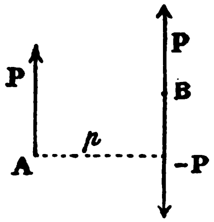

FIG. 33.

Hence, the force P acting at A is equivalent to a like parallel force P acting at B together with a couple.

59. **When any number of coplanar forces not in equilibrium act on a rigid body, they are equivalent either to a single force or a single couple.**

Let a number of forces P_1, P_2, P_3, and P_4 (we here take four forces, but the proof will equally apply for any number) act on a rigid body.

Now P_1 and P_2, if they do not themselves constitute a couple, are equivalent to a single resultant force: let this resultant be R_1.

Similarly let the resultant of R_1 and $P_3 = R_2$.

Hence there are only two forces left, viz. R_2 and P_4.

Now, either R_2 and P_4 are equal and unlike parallel forces, or they are not.

In the former case they constitute a couple, and in the latter case they are equivalent to a single resultant force.

NOTE.—If P_1 and P_2, or any other pair of forces, constituted in themselves a couple, then if the remaining forces were equivalent to a single force we could then have left a single force and a couple whose combined effect can be easily proved to be equivalent to a single force (converse of Art. 58). On the other hand, if the remaining forces were equivalent to a couple, we would have left two couples which are equivalent to a single couple (Art. 57).

NOTE.—The following example will serve as an illustration, showing how a force and a couple are equivalent to a single force.

Ex. Two forces, each of 10 lbs. wt., act from A to B and from C to D respectively along the sides AB, CD of a square $ABCD$; a third force of 15 lbs. wt. acts along the diagonal AC. Find their resultant.

If a be the side of the square, then the couple consisting of the forces 10, 10, acting at the extremities of the arm a, may be replaced by another couple of equal moment, each of whose forces is equal to that along AC, viz. 15, and whose arm is $\frac{2a}{3}$; for the moment of this couple, viz. $15 \times \frac{2a}{3}$ or $10a$, remains unchanged. Now, since a couple may be shifted anywhere in its own plane without altering its effect, the couple (15, 15) can be moved round into such a position that one of its 15 lbs. forces is opposite to the 15 lbs. force along AC, these two forces therefore neutralizing one another. There now remains a single resultant force equal to 15 lbs. wt. whose line of action is parallel to AC, and whose perpendicular distance from A is $\frac{2a}{3}$.

EXAMPLES VII.

1. Prove that a force and a couple are equivalent to a single force.

2. Find the resultant of a force of 6 lbs. wt. and a couple each of whose forces is 4 lbs. wt. and whose arm is 3 feet long; and state its position with reference to the 6 lbs. force.

3. Two forces, each of 10 lbs. wt., act along the sides AB and CD of a square $ABCD$ whose side is 6 feet long, and along AD and CB forces each equal to 4 lbs. wt. At A and C, forces each equal to $6\sqrt{2}$ lbs. wt. act, and whose directions are parallel to BD and DB respectively. Find the moment of the couple to which they are equivalent.

4. Three forces act along the sides of a triangle taken in order, each force being represented in magnitude by the side along which it acts: prove that they are equivalent to a couple whose moment is represented by twice the area of the triangle.

5. Prove also that, if forces act along the sides of a plane polygon taken in order, each being represented in magnitude by the side along which it acts, they are equivalent to a couple whose moment is twice the area of the polygon.

6. Find the resultant of the following forces whose components, parallel to two rectangular axes, are

$$10.\ 13;\ -7,\ 15;\ 11,\ -9;\ -9,\ -7,$$

and which act at the points whose coordinates are

$$5,\ 12;\ 3,\ -7;\ 4,\ -11;\ -2,\ 10.$$

7. Forces $(6, -4)$, $(-8, 3)$, $(5, -3)$, $(-11, -2)$ act at the points

$$(-5, 7),\ (3, -6),\ (-4, -7),\ (1, 5)$$

respectively: find the magnitude and position of the resultant.

8. Forces of magnitudes 18, 27, 15, 24 act in order along the sides of a square of area 16 square inches: are they equivalent to a couple? and, if not, find the magnitude, direction, and position of their resultant.

9. A weightless horizontal bar *AB* is acted upon by a vertical downward force of 1 lb. wt. at *A* and an equal vertical upward force at *B*, and a downward force of 5 lbs. wt. at a given point *C*, this force being inclined to the bar at an angle of 30°. Find at what point in the bar a force must be applied to balance these, and find also its magnitude and direction.

CHAPTER VI.

CENTRE OF GRAVITY.

60. **Centre of a System of Parallel Forces.**—We have seen (Arts. 48, 49) that the line of action of the resultant of two parallel forces cuts the line joining the points of application of the forces in a point which divides this line inversely as the forces, and that the position of this point depends solely on the magnitudes of the forces and the positions of their points of application, and will therefore be unaltered when the directions of the forces are turned at their points of application through a common angle This point is called the **centre** of the two parallel forces.

To find the Resultant of any number of parallel forces acting at fixed points of a rigid body.

61. Let parallel forces P, Q, S, and T act at A, B, C, and D respectively. It is required to find their resultant.

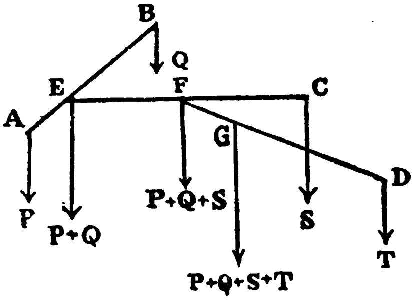

FIG. 34.

Join AB, and divide AB at E inversely as the forces P and Q; i.e.

$$AE : EB :: Q : P;$$

therefore, the resultant of P and Q, viz. $P + Q$, acts at E.

Now, join EC, and divide EC in F, so that

$$EF : FC :: S : P + Q.$$

Hence, the resultant of $P + Q$ and S, viz. $P + Q + S$, acts at F.

Again, join FD, and divide FD in G, so that

$$FG : GD :: T : P + Q + S.$$

Hence, the resultant of the four forces is their sum

$$P + Q + S + T,$$

acting at the point G, which is called the **centre** of the system of forces.

Similarly, we may find the resultant and centre of any number of parallel forces.

62. From the above construction, it is evident that the position of the point G depends solely on the magnitude of the forces and the positions of their points of application, but not on their directions; hence the following definition:—

Definition.—The centre of a system of parallel forces acting at fixed points of a rigid body is that point through which the line of action of the resultant will pass *whatever may be the direction* of the parallel forces.

Centre of Gravity.

63. The mass of every body may be considered as the sum of the masses of a great number of particles of matter; and, as the mass of each particle is attracted to the centre of the Earth, therefore what is called the *weight* of a body is really the resultant of all the forces due to the Earth's attraction on the different particles which compose it.

Also, if the magnitude of a body be small compared with that of the Earth, the lines of action of the forces with which its particles are attracted to the centre of the Earth may be regarded as parallel to one another; and therefore, the weights of the particles which compose the mass of a body may be regarded as a system of parallel forces, and the **centre** of this system is called the **centre of gravity** of the body.

Definition.—The centre of gravity of a body is that point through which the line of action of the resultant of the weights of its particles passes *whatever be the position of the body.*

64. It is to be borne in mind that the centre of gravity of a body need not necessarily be a point situated in the substance of the body itself; for example, the centre of gravity of a circular ring or of a hollow sphere is, in each case, at the geometrical centre, a point not situated in the actual substance of the body. In all cases, however, in which the centre of gravity of a body is situated within the substance of the body, it is evident that, if the body be supported at its centre of gravity, it will remain in equilibrium *in any position.*

65. **Every body has a centre of gravity.**—The proof of this proposition is the same as the proof that every system of parallel forces has a **centre** through which the resultant passes (Art. 61), the forces P, Q, S, T, etc., in this case being taken to represent the weights of the different particles which compose the body.

66. **A body cannot have more than one centre of gravity.**—For, suppose a body could have two centres of gravity, G and H, therefore, according to the above definition, the weight of the body (i.e. the resultant of the weights of all the particles which compose it) must pass through both G and H *in every position of the body.* This would certainly be possible if the body were held in such a position that the line GH was vertical (for the weight of a body acts vertically downwards), but could not be possible if the line GH were, for example, horizontal: for the weight of the body could not then pass through both G and H at the

same time. Therefore a body could not have more than one centre of gravity.

NOTE.—By the centre of gravity of an *area* is meant the centre of gravity of a uniform thin sheet or lamina, such as a piece of paper or tinfoil covering that area.

It is evident that the centre of gravity of a uniform straight rod or bar is at its middle point. This is sometimes expressed by saying that the centre of gravity of a straight line is its middle point.

To find the Centre of Gravity of a Parallelogram.

67. This means to find the centre of gravity of a uniform thin lamina in the form of a parallelogram.

Let $ABCD$ be the parallelogram (fig. 35), and let E and F be the middle points of the sides AB and CD respectively.

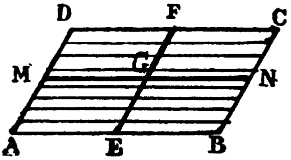

FIG. 35.

Now, imagine the parallelogram to be divided into a great number of thin strips parallel to the side AB; then the centre of gravity of each strip, being its middle point, the centre of gravity of the whole parallelogram must lie on the line EF which bisects each of the strips.

Similarly, it lies on MN, which joins the middle points M and N of AD and BC respectively.

Therefore, it must be at G, where EF and MN intersect; and it is evident that this point is also the point where the diagonals intersect.

To find the Centre of Gravity of a Triangle, that is, of a Triangular Lamina.

68. Let ABC (fig. 36) be the triangle, and let L and M be the middle points of the sides BC and CA respectively.

Imagine the triangle divided up into a great number of thin strips parallel to the sides BC. Join AL and BM.

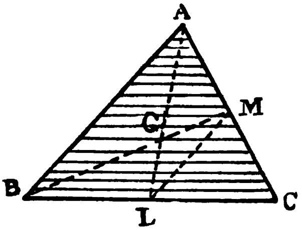

FIG. 36.

As before, the centre of gravity of each strip is its middle point. But the median AL which bisects BC, bisects each parallel to BC, and therefore AL passes through the centre of gravity of each strip.

Hence the centre of gravity of the triangle lies on AL.

Similarly it lies on BM.

Therefore the centre of gravity of the triangle is at the point G, viz. *the point of intersection of the medians.*

69. **The centre of gravity of a triangle divides each median in the ratio 2 : 1.**—To show this, join LM. Then LM is parallel to BA (Euclid VI, 2): therefore the triangles ABC and MLC are equiangular, and hence their sides are proportional (VI, 4);

$$\therefore \quad AB : BC :: ML : LC.$$

Alternately, $\quad AB : ML :: BC : LC;$

but $\quad BC : LC :: 2 : 1;$

$$\therefore \quad AB : ML :: 2 : 1.$$

Again the triangles AGB and LGM are equiangular

$\therefore \quad AB : BG :: LM : MG.$

Alternately, $\quad AB : LM :: BG : MG;$

but $\quad AB : LM :: 2 : 1;$

$\therefore \quad BG : GM :: 2 : 1.$

Hence *the centre of gravity of a triangle is at the point of intersection of the medians situated at a distance of one-third of each median from the middle point of each side or two-thirds of the median from each vertex.*

70. **The centre of gravity of a triangle coincides with the centre of gravity of any three equal particles placed at its vertices.**

For let three weights, each $= w$, be placed at the vertices A, B, and C of the triangle ABC (fig. 36).

Then w at B and w at C are equivalent to $2w$ at L, the middle point of BC.

Again, $2w$ at L and w at A are equivalent to $3w$ acting at a point G on AL, such that

$$AG : GL :: 2w : w \text{ or } 2 : 1 \quad \text{(Art. 48)};$$

but this is the same point as the centre of gravity of the triangle (Art. 69).

71. Conversely, **if a triangular slab be supported at its vertices, the pressure on each prop is the same, each being equal to one-third of the weight of the triangle.**

For let W be the weight of the triangle which acts at G, the centre of gravity of the triangle (fig. 36).

Then, since $AG : GL :: 2 : 1$, W acting at G may therefore be replaced by $\frac{2}{3} W$ at L and $\frac{1}{3} W$ at A.

Again, since L is the middle point of BC, $\frac{2}{3} W$ at L may be replaced by $\frac{1}{3} W$ at B and $\frac{1}{3} W$ at C. Hence the pressures on the supports A, B, and C are each one-third the weight of the triangle.

72. **To find the Centre of Gravity of a Polygon.**— Let $ABCDE$ (fig. 37) be the polygon. Divide it up into triangles by joining A to each of the other vertices.

Let L, M, and N be the respective centres of gravity of the triangles.

Join LM, and divide it in K, so that

$$LK : MK :: \text{wt. of } \triangle\ ACD : \text{wt. of } \triangle\ ABC.$$

Hence K is the centre of gravity of the quadrilateral $ABCD$; and we can imagine the sum of the weights of the two triangles to act at K.

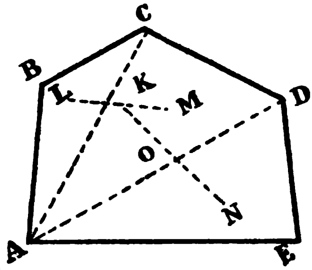

FIG. 37.

Now join KN, and divide it at O, so that
$KO : ON :: \text{wt. of } \triangle\ AED : \text{sum of wts. of } \triangle\text{s } ABC \text{ and } ACD$, and the point O is the centre of gravity of the five-sided polygon $ABCDE$, and similarly it may be found for a polygon with any number of sides.

Similarly, if we are given the weights and centres of gravity of the different parts which compose a body of any shape, we can find the centre of gravity of the whole body.

To find the Centre of Gravity of a number of Particles of given Weights placed at given Points in a Straight Line.

73. Let particles of weights w_1, w_2, w_3, w_4, etc., be placed at fixed points A, B, C, D, etc., in a straight line (fig. 38), and let the distances of A, B, C, D, etc., from a fixed point O in the same line be x_1, x_2, x_3, x_4, etc., respectively.

It is required to find the distance of the centre of gravity of the particles from O.

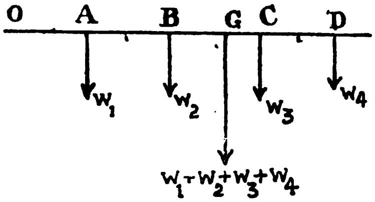

FIG. 38.

Let the centre of gravity of the particles be at G, and $\bar{x}$ represent the distance of G from O.

Now the resultant of the weights is equal to their sum

$$w_1 + w_2 + w_3 + w_4 + \ldots$$

Hence the moment of the resultant round O equals

$$(w_1 + w_2 + w_3 + w_4 + \ldots)\bar{x}.$$

Also the algebraic sum of the moments of the separate forces equals $w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + \ldots$

But the moment of resultant equals the algebraic sum of moments of forces (Art. 51);

$$\therefore (w_1 + w_2 + w_3 + w_4 + \ldots)\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + \ldots:$$

$$\therefore \quad \bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + \ldots}{w_1 + w_2 + w_3 + w_4 + \ldots}.$$

This is generally expressed thus:

$$\bar{x} = \frac{\Sigma wx}{\Sigma w}.$$

Ex. 1. Particles of 1, 2, 3, 4, 5, 6, and 7 lbs. wt. are hung at points in a straight line at distances 1, 2, 3, 4, 5, 6, and 7 feet respectively from a point O on the line: find how far from O is their centre of gravity situated.

Here the resultant R equals

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 \text{ lbs. wt.},$$

and let $\bar{x}$ equal the distance of the point of application of R from O; therefore, taking moments round the point O, we have

$$28\bar{x} = 1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 + 5 \times 5 + 6 \times 6 + 7 \times 7,$$

or $$28\bar{x} = 140; \quad \therefore \quad \bar{x} = 5 \text{ feet from } O.$$

Ex. 2. A uniform straight bar, 8 feet long, whose weight is 60 lbs., is divided into eight equal parts marked 0, 1, 2, 3, 4, 5, 6, 7, 8 (fig. 39), commencing at one extremity. At 0, 1, and 7 are hung respectively weights of 20, 30, and 50 lbs., and at 3 and 5 are applied upward thrusts of 20 lbs. and 40 lbs. wt. respectively. Find the centre of the whole system of parallel forces (including the weight of the bar) acting on the bar, i.e. find at what point the bar will balance in any position. The bar being uniform its weight may be supposed to act at the centre.

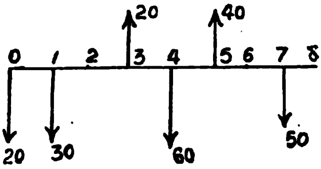

FIG. 39.

Here $R = 20 + 30 + 60 + 50 - 20 - 40 = 100$ lbs. wt.

Let $\bar{x}$ equal the distance of the point of application of R from O; therefore, taking moments round O, we have

$$100\bar{x} = 20 \times 0 + 30 \times 1 + 60 \times 4 + 50 \times 7 - 20 \times 3 - 40 \times 5 = 360;$$

$$\therefore \quad \bar{x} = 3\tfrac{3}{5} \text{ feet from } O.$$

Ex. 3. On a side of a square, of length 10 feet, an isosceles triangle is constructed of height 12 feet. Find the position of their common centre of gravity.

Here the area of the square = 100, and area of the triangle

$$= \tfrac{1}{2} 10 \times 12 = 60;$$

and, as their weights are proportional to their areas, therefore let their weights be 100 and 60 units acting at L and M, the centres of gravity of the square and triangle respectively (fig. 40), KM being = 4 feet, and KL = 5 feet.

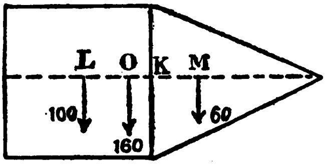

FIG. 40.

Now $R = 100 + 60 = 160$, and let it act at O, and let $LO = \bar{x}$.

Taking moments round L we have

$$160\bar{x} = 100 \times 0 + 60 \times 9, \quad \text{or} \quad 160\bar{x} = 540, \quad \text{or} \quad \bar{x} = 3\tfrac{3}{8} \text{ feet}:$$

therefore the centre of gravity is $3\tfrac{3}{8}$ feet from L, the centre of the square.

EXAMPLES VIII.

1. Prove that if a number of triangles stand on the same base and between the same parallels their centres of gravity lie on a line parallel to the base.

2. Given the base of a triangle in magnitude and position, and that the locus of its vertex is a straight line, show that the locus of its centre of gravity is also a straight line.

3. A heavy triangle ABC rests on the ground; and it is found that a certain vertical force applied at A is just sufficient to raise that vertex off the ground: show that the same force applied at B or C will have the same effect.

4. A telescope consists of two tubes, each of length 8 inches: when fully drawn out, it balances at a point 2 inches from the middle point. Compare the weights of the two tubes.

5. Prove that the centre of gravity of a triangle is the same point as the centre of gravity of the triangle formed by joining the middle points of its sides.

6. Parallel forces, whose magnitudes are 11, 12, 13, 14, and 15 lbs. wt., act at points in a line whose distances from a fixed point O in the line are 4, 5, 6, 7, and 8 feet respectively—the first and third force being opposite in direction to the others. Find the centre of the system.

7. Find the centre of gravity of a T-square made of two rectangular pieces of lengths 8 and 4 inches, and of breadths 2 and $2\frac{1}{2}$ inches respectively—the cross-piece being the shorter of the two.

8. A bar AB of length 10 feet, and whose weight is 3 lbs., has weights of 4 and 5 lbs. attached to the ends A and B respectively; if it balances at a point 6 feet from A, find the position of the centre of gravity of the bar.

9. Two circles whose radii are 4 and 2 inches are placed side by side, touching one another: find their common centre of gravity.

10. In Ex. 9, if the circles be replaced by homogeneous spheres of the same radii as the circles, find the centre of gravity.

11. Show that the centre of gravity of the perimeter of a triangle (e.g. a wire bent into the form of a triangle) is the centre of the circle inscribed in the triangle formed by joining the middle points of its sides.

12. If D be the middle point of the base BC of a triangle ABC, show that the distance between the centres of gravity of the triangles ABD and ACD is $\frac{1}{3}BC$.

The Centre of Gravity of a Body being given, and also the Centre of Gravity of a portion of the Body, to find the Centre of Gravity of the remaining portion.

74. Let A be the centre of gravity of the whole body whose weight is W (fig. 41), and B the centre of gravity of a portion whose weight is W': it is required to find the centre of gravity of the remaining portion.

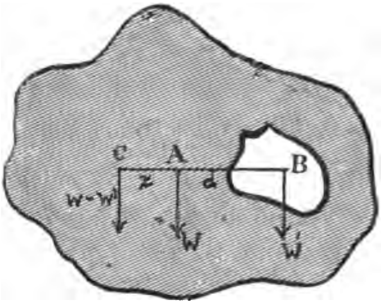

FIG. 41.

The weight of the remaining portion must equal $W - W'$.

Let C be the centre of gravity of this portion.

Hence the point A is the centre of gravity of the weights $W - W'$ acting at C, and W' acting at B; therefore the points C, A, and B are in the same straight line.

Also, if $AB = d$ and $AC = x$, we have

$$(W - W')x = W'd; \quad \therefore \quad x = \frac{W'd}{W - W'}.$$

Hence C lies on BA produced at a distance x from A given by the above equation.

Ex. Out of a uniform circular disk of radius 4 inches another circular disk of radius 1 inch is cut, the centres of the circles being 2 inches apart: find the centre of gravity of the remainder.

Since the area of a circle of radius $r = \pi r^2$;

$$\therefore \quad \frac{\text{area of whole circle}}{\text{area of part cut out}} = \frac{\pi 4^2}{\pi 1^2} = \frac{16}{1}.$$

Their weights, being proportional to their areas, may therefore be taken as 16 and 1 respectively, acting at their centres A and B; therefore the weight of

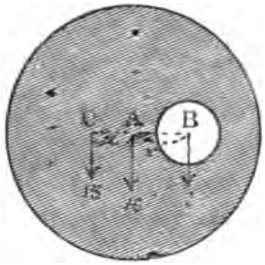

FIG. 42.

the remaining portion = 15. Let it act at C, and, if length of $AC = x$, we have

$$15x = 1 \times 2; \quad \therefore \quad x = \frac{2}{15} \text{ of an inch from } A,$$

the centre of the whole circle, and on the line BA produced.

75. We have just shown (Art. 73) how to find the distance of the centre of gravity of a number of particles placed in a line from any fixed point in the line. Now, if the particles are situated in one plane, but not in a line, we can still find their centre of gravity by finding its distances from any two axes at right angles to one another situated in the same plane as the particles.

76. To find the centre of gravity of a number of particles of given weight placed at given points in a plane.

Let the weights of the particles be w_1, w_2, w_3, etc., the particles being placed respectively at the points A, B, C, etc. (fig. 43).

Take any two straight lines OX and OY at right angles to one another in the same plane as A, B, C, etc.

Let x_1, x_2, x_3, etc., be the respective distances of A, B, C, etc., from OY, and y_1, y_2, y_3, etc., be the respective distances of A, B, C, etc., from OX.

Let G be the centre of gravity of the particles, and let the distances of G from OY and OX be $\bar{x}$ and $\bar{y}$ respectively.

Now the resultant $R = w_1 + w_2 + w_3$, etc., . . . acting at G.

Now imagine the plane held upright in such a position that OY is vertical.

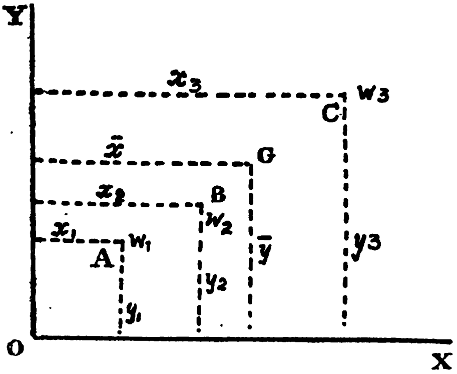

FIG. 43.

Then take moments round O (remembering that since OY is vertical, the weights all act parallel to OY, and therefore the respective distances of O from the lines of action of the weights are x_1, x_2, x_3, etc.);

$$\therefore \quad (w_1 + w_2 + w_3 + \ldots)\,\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 + \ldots;$$

$$\therefore \qquad \bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots} = \frac{\Sigma wx}{\Sigma w}.$$

Similarly, imagine the plane to be held upright with OX vertical; then the lines of action of the weights, being parallel to OX, the distances of O from their lines of action are y_1, y_2, y_3 respectively;

$$\therefore \quad (w_1 + w_2 + w_3 + \ldots)\,\bar{y} = w_1y_1 + w_2y_2 + w_3y_3 + \ldots;$$

$$\therefore \qquad \bar{y} = \frac{w_1y_1 + w_2y_2 + w_3y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots} = \frac{\Sigma wy}{\Sigma w};$$

therefore, since $\bar{x}$ and $\bar{y}$ are found, the position of G is determined.

Ex. 1. Particles of weights 4, 5, 6, and 7 lbs. are placed at the angular points *A*, *B*, *C*, and *D* respectively of a weightless square lamina whose side is 10 inches: find the position of the centre of gravity.

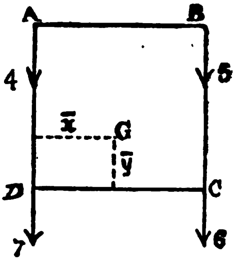

FIG. 44.

Here $R = 4 + 5 + 6 + 7$ lbs. wt.

Let it act at *G*, whose distances from *AD* and *DC* are $\bar{x}$ and $\bar{y}$ respectively.

Now imagine the square held so that *AD* is vertical, and take moments round *D* (remembering that, as the lines of action of 4 and 7 now pass through *D*, each of their moments round *D* is zero).

The moment of resultant = sum of moments of forces:

$$\therefore \quad (4 + 5 + 6 + 7)\bar{x} = 4 \times 0 + 5 \times 10 + 6 \times 10 + 7 \times 0,$$

$$22\bar{x} = 110; \quad \therefore \quad \bar{x} = 5 \text{ inches.}$$

Now imagine the square held so that *CD* is vertical, and take moments round *D* (remembering that, as the lines of action of 6 and 7 now pass through *D*, each of their moments round *D* is zero):

$$\therefore \quad (4 + 5 + 6 + 7)\bar{y} = 4 \times 10 + 5 \times 10 + 6 \times 0 + 7 \times 0,$$

$$22\bar{y} = 90; \quad \therefore \quad \bar{y} = 4\tfrac{1}{11} \text{ inches.}$$

Hence the centre of gravity is situated 5 inches from *AD*, and $4\frac{1}{11}$ inches from *DC*.

Ex. 2. If in Ex. 1 the square lamina had weighed 8 lbs., find at what point it would have balanced, the same particles as before being placed at the corners.

Taking the weight of the square to act at its centre, we obtain on taking moments, as before

$$(4 + 5 + 6 + 7 + 8)\bar{x} = 4 \times 0 + 5 \times 10 + 6 \times 10 + 7 \times 0 + 8 \times 5,$$

or

$$30\bar{x} = 150, \quad \text{or} \quad \bar{x} = 5 \text{ inches.}$$

$$(4 + 5 + 6 + 7 + 8)\bar{y} = 4 \times 10 + 5 \times 10 + 6 \times 0 + 7 \times 0 + 8 \times 5,$$

$$30\bar{y} = 130, \quad \text{or} \quad \bar{y} = 4\tfrac{1}{3} \text{ inches.}$$

Ex. 3. A letter *F* is formed of straight pieces of uniform wire of lengths 8, 4, 2, 2, 2, 2, and 2 inches: find the position of its centre of gravity.

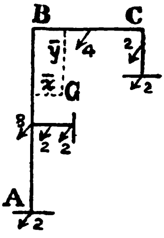

FIG. 45.

Since the weights of the pieces of wire are proportional to their lengths, let their weights be 8, 4, 2, 2, 2, 2, 2 units of weight respectively acting at their middle points.

Let the resultant weight, which is equal to their sum, act at *G* at distances $\bar{x}$ and $\bar{y}$ from *AB* and *BC* respectively.

Now, holding the *F* so that *AB* is vertical, and taking moments round *B*, we obtain

$$8 + 4 + 2 + 2 + 2 + 2 + 2)\bar{x} = 8 \times 0 + 4 \times 2 + 2 \times 4 + 2 \times 4 + 2 \times 1 + 2 \times 2 + 2 \times 0,$$

$$22\bar{x} = 30; \quad \therefore \quad \bar{x} = 1\tfrac{4}{11}.$$

Similarly, holding the F so that BC may be vertical, and taking moments round B,

$$22\bar{y} = 8 \times 4 + 4 \times 0 + 2 \times 1 + 2 \times 2 + 2 \times 4 + 2 \times 4 + 2 \times 8,$$

$$22\bar{y} = 70; \quad \therefore \quad \bar{y} = 3\tfrac{2}{11} \text{ inches};$$

therefore the C.G. is situated $1\tfrac{4}{11}$ inches from AB, and $3\tfrac{2}{11}$ inches from BC.

Ex. 4. The cross-section of a girder consists of three rectangles, the lower 2 inches vertical by 16 inches horizontal, the middle 16 inches vertical by 2 inches horizontal, and the upper 2 inches vertical by 4 inches horizontal, their centres of gravity being in the same vertical line: find the centre of gravity of the whole section.

The respective areas of the rectangles being 32, 32, and 8 square inches, these numbers may be taken to represent the weights of the rectangles; and as these weights act at points in the same vertical line, therefore the C.G. of the whole section lies on this line.

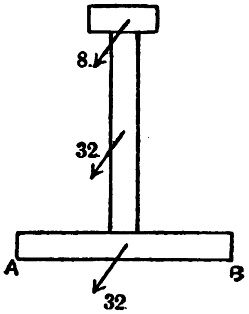

FIG. 46.

Let $\bar{y}$ = the height of the C.G. above AB;

$$\therefore \quad (32 + 32 + 8)\bar{y} = 32 \times 1 + 32 \times 10 + 8 \times 19,$$

or

$$72\bar{y} = 504; \quad \therefore \quad \bar{y} = 7 \text{ inches above } AB.$$

EXAMPLES IX.

1. From a uniform circular plate another circular plate, having for its diameter the radius of the first, is cut away: find the centre of gravity of the remainder.

2. Where must a circular hole of radius 6 inches be cut out of a circular disk of radius 2 feet in order that the centre of gravity of the remainder may be distant 1 inch from the centre of the disk?

3. The sides of a parallelogram are bisected, and the points of bisection of the opposite sides joined; if one of the four parallelograms so formed be cut out, find the centre of gravity of the remainder.

4. If from a parallelogram one of the four triangles formed by drawing the diagonals be cut away, find the centre of gravity of the remainder.

5. A triangle is cut off a parallelogram by a section made through the middle points of two adjacent sides: find the distance of the centre of gravity of the remainder from the centre of the parallelogram.

6. From a uniform triangular lamina is cut the triangular portion formed by joining the middle points of two sides: find the centre of gravity of the remaining portion.

7. A parallel DE is drawn to the base BC of a triangle ABC such that $AD = \frac{1}{3}AB$, and the triangle ADE is cut away: find the centre of gravity of the remainder.

8. If a and b are the two parallel sides of a trapezium, show that the centre of gravity divides the line joining their middle points in the ratio

$$2b + a : 2a + b.$$

9. Prove that the centre of gravity of three weights placed at the vertices of a triangle, each weight being proportional to the opposite side, is the centre of the inscribed circle of the triangle.

10. Masses of 1, 2, 3, and 4 lbs. are placed at the angular points A, B, C, D respectively of a horizontal square lamina which weighs 5 lbs.: find the position of the common centre of gravity.

11. Three masses weighing 1, 2, and 4 lbs. are placed at the vertices A, B, and C respectively of an equilateral triangle ABC: find the distance of their centre of gravity from A.

12. The sides BC, CA, and AB of a triangle are of lengths 3, 4, and 5 feet respectively. Weights of 6, 9, and 12 lbs. are placed respectively at A, B, and C: find the position of their centre of gravity.

13. Find the centre of gravity of a letter K made of pieces of the same wire whose respective lengths are 8, 5, 5, 2, 2, 2, and 2 inches.

14. Three circles of radii 2, 3, and 10 inches are made of the same wire, and soldered together so that each touches the other two, and the three lie in the same plane: find the distance of their centre of gravity from the sides of the triangle formed by their centres.

15. Find the centre of gravity of the figure formed by three rectangles whose breadths are 17, 1, and 46, and heights 3, 19, and 2 respectively, and which are placed on the top of each other with their centres in the same vertical line.

Properties of the Centre of Gravity.

77. **If a body be suspended at any point, it will settle in equilibrium in that position in which its centre of gravity is in the vertical line drawn through the point of suspension.**

For, let *O* be the point at which the body is suspended (fig. 47), and *G* the centre of gravity of the body.

Then, when the body has settled into a state of rest, we may regard it as in equilibrium under the action of two forces:—

(1) Its weight *W*, acting vertically downwards through its centre of gravity *G*.

(2) The reaction *R*, acting through the fixed point *O* at which the body is supported.

But when a body is in equilibrium under the action of two forces, these forces must be equal and opposite, and their lines of action must be in the same straight line: hence the points *O* and *G* are in the same vertical line.

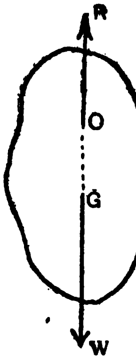

FIG. 47.

78. This proposition is true, not only when the point *O* is above the centre of gravity of the body, as when the body is suspended by a string attached to the point *O* (fig. 47), but also when the point *O* is below the centre of gravity: as, for example, when a billiard-ball rests on a horizontal table the point at which the ball is in contact with the table is vertically below the C.G. of the ball.

To find experimentally the Centre of Gravity of a Plane Body or Lamina of any Shape.

79. Suspend the body by any point, and, when it settles into a position of equilibrium, draw a vertical through the point of suspension. The centre of gravity of the body must lie on this vertical line (Art. 77).

Again, suspend the body by any other point, and, when in equilibrium, draw another vertical through the point of suspension (the two points being so chosen that the two verticals shall not coincide). The centre of gravity of the body must also lie on this vertical line.

Therefore the point of intersection of the two verticals is the centre of gravity of the body.

NOTE.—This method can be applied *practically* only to finding the C.G. of a lamina such as a piece of paper or a bent wire all of whose parts are in one plane. It is certainly *theoretically* applicable to all bodies; but, in applying it to find the C.G. of such bodies as a chair, or a table, or a lump of stone, it would be very difficult to mark the directions of the verticals.

80. **Definition.**—If a body rests on any plane surface, and a thread be drawn tightly round it so as to enclose all points of the body in contact with the surface, the area enclosed by the thread is called the **base.**

Thus, if the undotted line (fig. 48) represents the actual *geometric* base of the body, and if the dotted line *ABCDEF* represents a thread stretched round it, then the area *ABCDEF* marked out by the thread is the **base.**

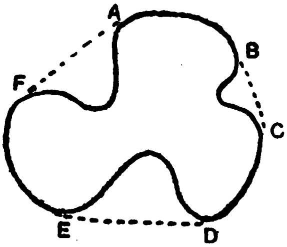

FIG. 48.

81. **When a body is placed on a plane, it will stand or topple over according as the vertical drawn through its centre of gravity falls within or without the base.**

This proposition is true whether the plane on which the body is placed be horizontal or inclined; in the case of an inclined plane we must, however, suppose that there is sufficient friction to prevent the body from sliding down the plane.

For, let *ABC* represent a cone or other body placed on a plane *MN*, and let the vertical drawn through its centre of gravity *G* fall within the base *BC* (fig. 49*a*), or outside the base (fig. 49*b*).

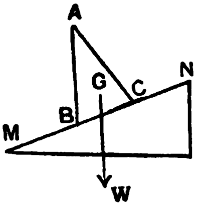

FIG. 49 (*a*).

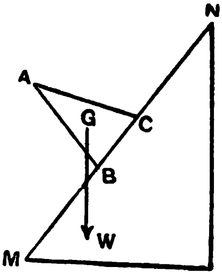

FIG. 49 (*b*).

Now, in either case, the normal or perpendicular reactions of the plane against the different points of the base of the body, and the forces, due to friction (if the plane be not horizontal), acting at these points, may be compounded into a single resultant reaction which must act at some point *within the base BC*. That this resultant reaction may be in equilibrium with the weight *W* acting through *G*, the vertical through *G* must also fall within *BC* as in fig. 49 (*a*). But if the vertical through *G* falls outside the base (fig. 49 (*b*)), the body will therefore topple over.

Stable, Unstable, and Neutral Equilibrium.

82. **Definition (1).**—A body is said to be in **stable** equilibrium when, if it be slightly displaced, it tends to return to its original position of equilibrium.

Definition (2).—A body is said to be in **unstable** equilibrium when, if it be slightly displaced, it tends to deviate further from its original position of equilibrium.

Definition (3).—A body is said to be in **neutral** equilibrium when, if it be slightly displaced, it remains in its new position.

Examples of *Stable Equilibrium.*—A marble in the bottom of a cup or hemispherical bowl, a swing, a pendulum, any body suspended by a point above its centre of gravity, or a cone placed with its base on a horizontal table.

Examples of *Unstable Equilibrium.*—A marble balanced on the top of a sphere, a golf-ball balanced on a finely pointed tee, a cone balanced on its vertex on a horizontal table, or any body balanced on a point below its centre of gravity, and all such bodies as are commonly called "top-heavy."

Examples of *Neutral Equilibrium.*—A sphere floating in a fluid, a body suspended at its centre of gravity, or, in the case of rolling displacements, a billiard-ball resting on a horizontal table, or a cone resting on its side on the table.

EXAMPLES X.

1. A square table stands on four legs situated respectively at the middle points of the sides: find the greatest weight that can be placed at one of the corners without upsetting the table.

2. A circular table has three legs at equal distances round its circumference: find the greatest weight that can be placed at a point diametrically opposite one leg without upsetting the table.

3. A triangular lamina is suspended by a string attached to one of its vertices: if, when in equilibrium, the base is horizontal, prove that the triangle is isosceles.

4. A solid equilateral prism is placed with its axis horizontal on an inclined plane: if it be prevented from slipping, and the inclination of the plane be gradually increased, find the angle of inclination when the prism will just topple over.

5. If three forces meet in a point, prove that their resultant passes through the centre of gravity of three equal particles placed at the extremities of the lines representing the forces, and is represented in magnitude by three times the length of the line joining the point to the centre of gravity of the particles.

6. A cylinder, whose base is a circle 1 foot in diameter, and whose height is 3 feet, rests on a horizontal plane with its axis vertical. Find how high one edge of the base can be raised without causing the cylinder to topple over.

7. Two books which are of the same dimensions and mass are placed one on top of the other, so that their narrowest surfaces or thicknesses are in the same vertical plane, the length of the upper overlapping that of the lower, the lower book resting on a table so that the ends of the books are parallel to the edge of the table: show that the greatest amount of the upper book which can project beyond the edge of the table, consistent with equilibrium, is three-fourths of its length.

8. A heavy triangle ABC is suspended successively from the angles A and B, and the two positions of any side are found to be at right angles to each other: prove that $5c^2 = a^2 + b^2$.

CHAPTER VII.

CONDITIONS OF EQUILIBRIUM.

83. In this Chapter we shall discuss the equilibrium of a rigid body under the action of a number of forces whose lines of action lie in the same plane.

When three non-parallel forces act on a rigid body and keep it in equilibrium, their lines of action must meet in a point.

For, the three forces being P, Q, and R, let the lines of action of P and Q meet in a point O (fig. 9). The forces P and Q may therefore be supposed to act at O (axiom 1). Compound P and Q into a single resultant acting at O.

Now, since the three forces P, Q, and R are in equilibrium, the third force R must be equal and opposite to this resultant of P and Q, and must, therefore, also pass through O.

84. We thus see that one condition that three non-parallel forces should be in equilibrium is that their lines of action should be *concurrent*, i.e., meet in a point. By the application of the Triangle of Forces, or Lami's Theorem, or by resolution in two directions at right angles to one another, all the conditions of equilibrium can be obtained. In fact, three non-parallel forces are in equilibrium if

(1) their lines of action are concurrent, i.e. meet in a point,

(2) they are represented in magnitude and direction by the sides of a plane triangle taken in order.

Ex. 1. A particle whose weight is 24 lbs. hangs by a string, whose length is 13 inches, tied to a fixed point A: what horizontal force must be applied to the particle so as to keep it in such a position that its distance from a vertical drawn through A is 5 inches? Also find the tension of the string.

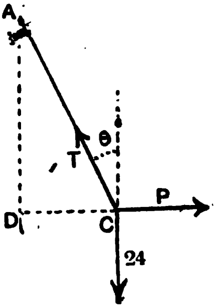

FIG. 50.

Let C be the position of the particle. Then $AC = 13$ inches and CD (the perpendicular from C on the vertical drawn through A) = 5 inches. $\therefore AD = 12$ inches.

Now the particle at C is in equilibrium under the action of three forces: (1) the horizontal force P; (2) the tension of the string T; (3) its own weight of 24 lbs. vertically downwards. Resolve these forces vertically (P having no vertical component).

$$\therefore T\cos\theta - 24 = 0, \quad \text{or} \quad \frac{T12}{13} = 24. \quad \therefore T = 26 \text{ lbs. wt.}$$

Again, resolve the forces horizontally (24 having no horizontal component),

$$\therefore P - T\sin\theta = 0, \quad \text{or} \quad P = T\sin\theta = 26 \times \frac{5}{13} = 10 \text{ lbs. wt.}$$

Otherwise thus:—Since the forces P, T, and 24, which are in equilibrium, are parallel to the sides of the triangle ADC taken in order, therefore they are proportional to those sides (Art. 26).

$$\therefore \frac{P}{DC} = \frac{T}{CA} = \frac{24}{AD} \quad \text{or} \quad \frac{P}{5} = \frac{T}{13} = \frac{24}{12};$$

$\therefore P = 10$, and $T = 26$ as before.

These results could also have been obtained by applying Lami's Theorem, or by taking moments.

Ex. 2. A mass of 20 lbs. wt. is suspended by two strings of length 3 feet and 4 feet, whose extremities are attached to two pegs in a horizontal line 5 feet apart: find the tensions of the strings.

Let AO and BO be the strings attached to the pegs A and B, AB being horizontal (fig. 51), and let the tensions of AO and BO be denoted by T and T' respectively.

Now, since $5^2 = 3^2 + 4^2$, $\therefore \angle AOB = 90°$ (Euc. I. 48).

The point O is in equilibrium under the action of the two tensions T, T', and the weight of 20 lbs. acting vertically downwards.

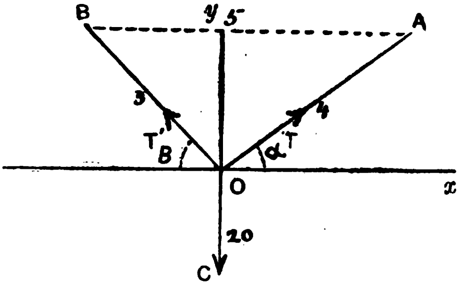

FIG. 51.

Draw horizontal and vertical lines Ox and Oy through O, and let α and β be the angles made by the strings with the horizontal line (fig. 51).

Resolving the forces horizontally and vertically, and equating each result to zero, we have :—

$$T\cos\alpha - T'\cos\beta = 0,$$

$$T\sin\alpha + T'\sin\beta - 20 = 0;$$

but $\cos\alpha = \cos A = \frac{4}{5}$, and $\cos\beta = \cos B = \frac{3}{5}$. Similarly for $\sin\alpha$ and $\sin\beta$.

$$\therefore \frac{T4}{5} - \frac{T'3}{5} = 0,$$

$$\frac{T3}{5} + \frac{T'4}{5} = 20;$$

solving for T and T' we get $T = 12$ lbs. wt., and $T' = 16$ lbs. wt.

Otherwise thus :—By Lami's Theorem, we have

$$\frac{T}{\sin BOC} = \frac{T'}{\sin COA} = \frac{20}{\sin AOB},$$

but $$\sin BOC = \sin BOy = \sin A = \frac{3}{5},$$

$$\sin COA = \sin AOy = \sin B = \frac{4}{5},$$

and $$\sin AOB = \sin 90 = 1.$$

$$\therefore \frac{T}{\frac{3}{5}} = \frac{20}{1} \quad \therefore \quad T = 12,$$

and $$\frac{T'}{\frac{4}{5}} = \frac{20}{1}, \quad \therefore T' = 16.$$

The same results could have been obtained by applying the principle of the Triangle of Forces: T, T', and 20, being each perpendicular to a side of the triangle AOB, are therefore proportional to those sides (Cor. 2, Art. 25).

Ex. 3. A mass weighing 56 lbs. is suspended to two strings of length 13 inches and 15 inches respectively, whose extremities are tied to two pegs in the same horizontal line 14 inches apart: find the tensions on the strings.

As before, take Ox and Oy drawn horizontally and vertically through O.

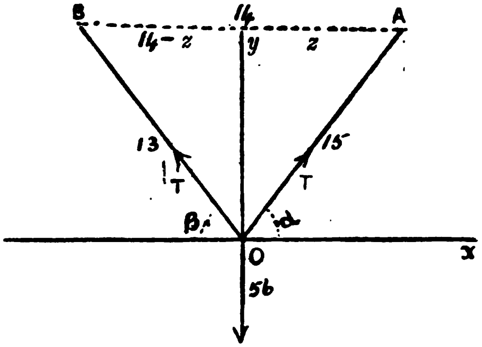

FIG. 52.

Let $Ay = z$, $\therefore$ $By = 14 - z$.

Now since $Oy^2 = 15^2 - z^2$, and also $Oy^2 = 13^2 - (14 - z)^2$,

$\therefore$ $15^2 - z^2 = 13^2 - (14 - z)^2$,

$$225 - z^2 = 169 - 196 + 28z - z^2, \quad \therefore z = 9 \text{ and } By = 5,$$

and hence $Oy = 12$.

Now, resolving along Ox and Oy in succession, we have

$$T\cos\alpha - T'\cos\beta = 0,$$

$$T\sin\alpha + T'\sin\beta - 56 = 0,$$

$$\text{or} \quad \frac{T9}{15} - \frac{T'5}{13} = 0,$$

$$\text{and} \quad \frac{T12}{15} + \frac{T'12}{13} = 56.$$

Solving for T and T' as before, we obtain $T = 25$ lbs. wt. and $T' = 39$ lbs. wt.

The same results could be obtained by the application of Lami's Theorem.

Ex. 4. A uniform rod AB of weight 10 lbs., hinged at A, is kept in a horizontal position by a string attached to B, and making an angle of 45° with the rod: find the tension of the string and the reaction of the hinge.

Let T be the tension of the string, and R the reaction of the hinge. Then the three forces keeping the rod in equilibrium are:—

(1) Its own weight, 10 lbs., acting vertically downwards through its middle point G.

(2) The tension of the string T acting along the string.

(3) The reaction R of the hinge (fig. 53).

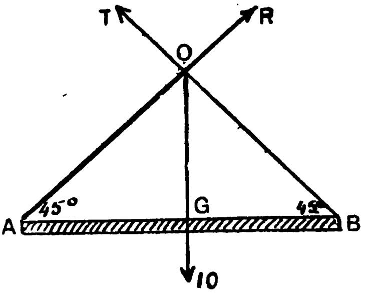

FIG. 53.

Now, the lines of action of three non-parallel forces which are in equilibrium must meet in a point: therefore the line of action of R must pass through O, the point of intersection of the vertical through G and the string.

Now, applying Lami's Theorem, it being evident from the diagram that the angles opposite T, R, and 10 are respectively 135°, 135°, and 90°, we have:—

$$\frac{T}{\sin 135^\circ} = \frac{R}{\sin 135^\circ} = \frac{10}{\sin 90^\circ},$$

$$\text{or} \qquad \frac{T}{\frac{1}{\sqrt{2}}} = \frac{R}{\frac{1}{\sqrt{2}}} = \frac{10}{1},$$

$$\therefore\ T = \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \quad \text{also} \quad R = 5\sqrt{2},$$

Otherwise thus:—Resolving the forces vertically we have

$$T \cos 45^\circ + R \cos 45^\circ - 10 = 0.$$

But $T = R$, since they make equal angles with the vertical,

$$\therefore\ 2T \cos 45 = 10, \text{ or, } \frac{2T}{\sqrt{2}} = 10, \quad \therefore\ T = R = 5\sqrt{2}.$$

NOTE.—It is important that the student should bear in mind the main principle involved in the above solution, viz., that when three forces are in equilibrium, and the lines of action of two of them meet in a point, the line of action of the third must pass through this point, and that, therefore, when the lines of action of two of the forces are known, that of the third can usually be determined.

Ex. 5. A uniform ladder *AB* of weight 480 lbs. and length 26 feet rests with one end *A* on the ground and the other end *B* against a smooth vertical wall *BC*; given the height of the ladder *BC* to be 24 feet, find the normal reaction of the wall and the reaction of the ground.

Evidently $AC^2 = 26^2 - 24^2 = 100$. $\therefore$ $AC = 10$ feet.

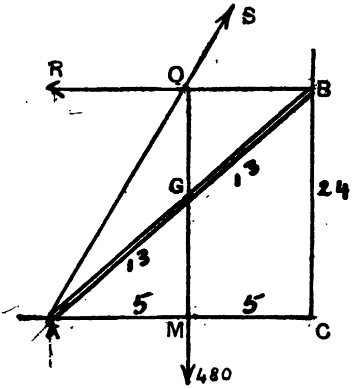

Fig. 54.

The three forces acting on the ladder are :—

(1) Its weight, 480 lbs., acting vertically downwards through *G*, its centre of gravity, which is its middle point (the ladder being uniform).

(2) The reaction *R* of the wall, which acts perpendicularly to the wall (the wall being smooth).

(3) The reaction *S* of the ground against the foot of the ladder *A*.

But since the vertical through *G* and the line of action of *R* meet at *O*, then the reaction *S* must pass through *O* (Art. 83).

Take moments round *A* (the moment of *S* round *A* being zero).

$$\therefore \quad R \times 24 - 480 \times 5 = 0. \quad \therefore \quad R = 100 \text{ lbs. wt.}$$

Also *S* is equal and opposite to the resultant of *R* and 480.

$$\therefore \quad S = \sqrt{100^2 + 480^2} = 20\sqrt{601} \text{ lbs. wt.}$$

Otherwise thus:—The lines of action of *R*, *S*, and 480 are parallel to the sides of the triangle *OMA*: hence they are proportional to those sides (Art. 26).

$$\therefore \quad \frac{R}{MA} = \frac{S}{AO} = \frac{480}{OM} \quad \text{and} \quad AO = \sqrt{5^2 + 24^2} = \sqrt{601}.$$

$$\therefore \quad \frac{R}{5} = \frac{S}{\sqrt{601}} = \frac{480}{24}.$$

$$\therefore \quad R = 100 \quad \text{and} \quad S = 20\sqrt{601} \text{ as before.}$$

To find the Conditions of Equilibrium of a number of Coplanar Forces acting on a Rigid Body.

85. We have already found the conditions under which any number of coplanar forces acting on a particle (and therefore whose lines of action meet in a point) should be in equilibrium. If, however, the forces are not concurrent, they will be in equilibrium :—

(1) **If the algebraic sums of their resolved parts along any two directions are separately zero.**

(2) **If the algebraic sum of their moments about any point in their plane is zero.**

NOTE.—In the first condition it is generally more convenient to choose the two directions at right angles to one another; but it is equally true for *any* two directions.

86. **Proof.**—For, if the forces are not in equilibrium, they reduce either to a single resultant force or a single couple (Art. 59).

But if the algebraic sums of the resolved parts along any two directions are separately zero, there cannot be a single resultant force (Art. 34).

Again, we know (Art. 54) that the algebraic sum of the moments of the forces of a couple about any point in their plane is a definite constant quantity, and hence could not equal zero, and therefore the forces could not reduce to a single couple.

Hence the system of forces must be in equilibrium.

The conditions of equilibrium of a number of coplanar forces acting on a rigid body may also be expressed as follows :—

87. **If the algebraic sums of the moments of the forces round any three points in their plane which are not in the same straight line are separately equal to zero, the system of forces is in equilibrium.**

For, let A, B, and C be the vertices of any triangle in the plane of the forces (A, B, and C being therefore not collinear).

Now, if the forces are not in equilibrium, they must reduce either to a single couple or to a single resultant force.

But, since the algebraic sum of the moments of the forces about A is zero, the forces cannot reduce to a single couple (Art. 54).

Hence they can only reduce to a single resultant force whose line of action must pass through A (Art. 47).

Similarly the line of action of this resultant must pass through the points B and C.

Hence the points A, B, and C are in the same straight line, which is contrary to the hypothesis.

Therefore the forces are in equilibrium.

EXAMPLES XI.

1. A particle whose weight is 16 lbs. hangs by a string, whose length is 10 inches, tied to a fixed point A: what horizontal force must be applied to the particle so as to keep it in such a position that its distance from a vertical line drawn through A is 6 inches? Also find the tension of the string.

2. A mass of 52 lbs. is suspended by two strings of lengths 5 feet and 12 feet, whose extremities are attached to two pegs in a horizontal line 13 feet apart: find the tensions of the strings.

3. If two strings, 9 inches and 40 inches long, support a weight of 82 lbs., the strings being fastened to two points in the same horizontal line at such a distance apart that the strings are perpendicular to each other, find the tensions in the strings.

4. A string of length $2l$ is fixed at its extremities to two points in the same horizontal line, whose distance apart is $2a$, and supports a smooth ring of weight W: find the tension of the string.

5. A string is fixed at its extremities to two points in the same horizontal line, and supports a smooth ring weighing 10 ozs.; the two parts of the string contain an angle of 60°: find the tension of the string.

6. A uniform rod AB, weighing 10 lbs., and capable of moving round one extremity B, which is fixed, is kept in a horizontal position by a string attached at A, which makes an angle of 30° with the rod: determine the tension of the string, and the reaction of the hinge at B.

7. A uniform rod AB, weighing 21 lbs., of length 3 feet, and capable of moving round one extremity B, which is fixed, is kept in a horizontal position by a string AC, which is attached to a point C situated 27 inches vertically over the hinge B: find the tension of the string, and the reaction of the hinge.

8. A cord is attached to a fixed point, and passes over a smooth pulley on the same level with the point, and 10 feet distant. A smooth ring of weight w slides on the cord; the cord is kept tight by a weight W attached to the free end: find the depth to which the ring will sink

$$(1) \text{ when } w = W, \quad (2) \text{ when } w = \frac{W}{3}.$$

9. A weight of 32 lbs. is suspended from the middle point of a string 20 inches long, whose extremities are attached to two points 1 foot apart in the same horizontal line: find the tension of the string.

10. A picture whose weight is 48 lbs. is suspended by means of a string 26 inches long passing over a smooth peg, and whose extremities are tied to two rings in the upper edge of the picture 10 inches apart: find the tension of the string.

11. A ladder, when placed against a wall of a house, just reaches a window. The ladder is now lowered by means of a rope attached to its topmost rung; the end of the rope being held by a person at the window, while the lower end of the ladder remains fixed at the angle where the wall and ground meet: find the tension of the rope when its (the rope's) direction makes an angle of 30° with the horizon.

12. Show that, if a ladder be placed with one end on the ground, supposed to be perfectly smooth, and the other end leaning against a smooth vertical wall, it could not remain in equilibrium.

13. A uniform beam rests at an inclination θ to the horizon against a smooth vertical wall, and on a smooth horizontal plane, and is prevented from slipping by a rope attached to its lower end, and to the foot of the wall; if W be the weight of the beam, prove that the tension of the rope is

$$\tfrac{1}{2} W \cot \theta.$$

14. A uniform ladder, of weight 288 lbs., and length 26 feet, rests against a smooth vertical wall, and on a smooth horizontal plane; it is kept in equilibrium by a rope attached to the lower end of the ladder, and to the foot of the wall. Given the distance of the lower end of the ladder from the foot of the wall to be 10 feet, find the tension of the rope.

15. If, in example 14, the rope were attached to the middle point of the ladder, and to a point in the wall half-way between the upper end of the ladder and the ground, find what would then be the tension of the rope.

16. A sphere of weight W rests on two inclined planes whose angles of inclination to the horizon are α and β: find the pressure on each plane.

17. A uniform rod AB, of weight W, can turn freely about the end A which is fixed: find what horizontal force applied at the end B will deflect the rod so that it makes (a) an angle of 45°, (b) an angle of 30° with the vertical.

18. A sphere rests between two horizontal bars which are in the same horizontal plane, the distance between the bars being equal to the radius of the sphere: prove that the reaction of each bar is

$$\frac{1}{3} W \sqrt{3}.$$

where W is the weight of the sphere.

19. A weight of 42 lbs. is supported by two strings of lengths 25 and 17 inches which are tied to two pegs in the same horizontal line at a distance of 28 inches apart: find the tensions in the strings, and the greatest horizontal force which can be applied to the weight without disturbing equilibrium.

20. A uniform rod AB, hinged at A, is pulled aside from the vertical by a horizontal force applied at B and equal to half the weight of the rod: at what inclination to the vertical will the rod rest?

21. ACB is a uniform rod, of weight W: it is supported (B being uppermost) with its end A against a smooth vertical wall AD by means of a string CD, DB being horizontal and CD inclined to the wall at an angle of 30°. Find the tension of the string and the pressure on the wall, and prove that

$$AC = \frac{1}{3} AB.$$

22. A mass of 1 cwt. is suspended by two strings of lengths $6\frac{1}{2}$ inches and $7\frac{1}{2}$ inches respectively, whose extremities are tied to two pegs in the same horizontal line 7 inches apart: find the tensions on the strings.

CHAPTER VIII.

MACHINES. THE LEVER, AND WHEEL AND AXLE.

88. By a **Machine** is meant any contrivance by means of which pressure or motion may be transmitted from one point to another, and altered both in magnitude and direction.

89. In all machines a force applied at one point is made to overcome a resistance acting at another point. The former force is called the **Power** or **Effort**, and the latter force the **Resistance** or **Weight**; and when one of these forces just balances the other, the two forces keep the machine in equilibrium. But it might be said that when two forces equilibrate each other, they must be equal and opposite, and that therefore the Power must be equal to the Resistance; and hence the machine could not enable us (as it often does) to overcome a very large Resistance by means of a small Power. This would certainly be true if the machine were free, i.e. not in contact with fixed points or surfaces. But as all machines contain fixed points, and in many cases fixed surfaces, which support part of the Resistance, it follows that by a proper arrangement of the position of these fixed points or surfaces the Power may be made to bear any ratio we choose to the Resistance or Weight.

In this Chapter we shall take no account of the friction between the different parts of the machine, and also, except when expressly mentioned, we shall neglect the weights of the different parts of the machine itself.

90. **Mechanical Advantage.**—In any machine the ratio of the Resistance to the Power is called the *mechanical advantage.* That is, if a power P balance a resistance or weight W, then—

$$\text{The Mechanical Advantage} = \frac{W}{P}.$$

91. **Mechanical Powers.**—The simplest forms of machines to which all other machines, no matter how complicated, can be reduced, are called the Mechanical Powers. They may be classified as follows:—

- I. The Lever, including, ...
  - 1. The Lever proper.
  - 2. The Wheel and Axle.
- II. The Cord, ...
  - 3. The Pulley.
- III. The Inclined Plane, including
  - 4. The Inclined Plane proper.
  - 5. The Wedge.
  - 6. The Screw.

The Lever.

92. **Definition.**—A Lever is a rigid bar capable of motion about a fixed point called the **Fulcrum.**

The **arms** of a lever are the perpendicular distances of the fulcrum from the lines of action of the power and the weight.

Levers are generally divided into Three Orders or Classes.

93. **First Order.**—Here the Fulcrum is between the Power and the Weight (fig. 55).

Examples:—A Poker when it is used to stir the fire, the point where it rests on the bar of the grate being the fulcrum: the Handle of a Pump; a Claw-Hammer when used to extract nails; a Pair of Scales; a Crowbar when a point within it rests on a fixed support and the extreme end is used to raise a heavy weight.

Examples of double levers of this order are a Pair of Scissors and a Pair of Pincers.

94. **Second Order.**—Here the weight is between the Power and the Fulcrum (fig. 56).

Examples :—A Wheelbarrow ; an Oar, if we assume the end of the oar in contact with the water to be at rest, the reaction of the rowlock against the oar being the Weight to be overcome ; a Crowbar, when the end rests on the ground.

An example of a double lever of this order is a Pair of Nutcrackers.

95. **Third Order.**—Here the Power is between the Resistance and the Fulcrum (fig. 57).

Examples :—The Treadle of a Lathe ; the Human Forearm, when extended supporting a weight on the hand, the elbow being the Fulcrum, and the force exerted by the contraction of the biceps muscle the Power.

Examples of double levers of this order are a pair of Sugar-tongs, or an ordinary Tongs, and the Forceps used in dissecting.

To find the Conditions of Equilibrium in the Lever.

96. The lever is here supposed to be a straight bar, and the directions of the power and weight to be perpendicular to the length of the lever.

Levers of the First Order.—Here P and W are two like parallel forces which are in equilibrium with the reaction R of the fulcrum at F (fig. 55);

$\therefore \quad R = P + W$ (Art. 48).

Also, if the lengths AF and BF be denoted by x and y respectively, we have, since the moments of P and W round F are equal and opposite,

$$\therefore \quad P \times x = W \times y.$$

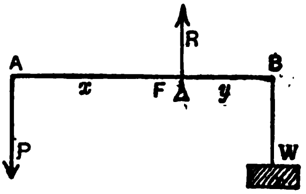

FIG. 55.

97. **Levers of the Second Order.**—Here P and W are unlike parallel forces, W being the greater (fig. 56). Hence

$R = W - P$ (Art. 49).

In this case also, if the lengths of the arms AF and BF be denoted by x and y respectively, we have $P \times x = W \times y$.

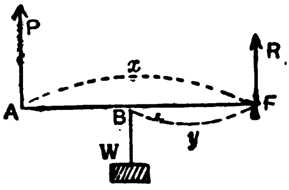

FIG. 56.

98. **Levers of the Third Order.**—Here P and W are also unlike parallel forces, P being the greater; hence

$$R = P - W.$$

Also, as before,

$$P \times x = W \times y.$$

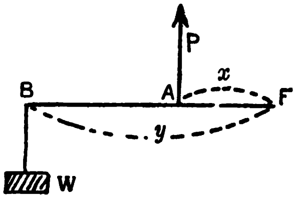

FIG. 57.

99. We have now shown that, in all three orders of lever,

$$P \times x = W \times y,$$

that is,

$$P \times \text{arm of } P = W \times \text{arm of } W,$$

or the Mechanical Advantage

$$\frac{W}{P} = \frac{x}{y} = \frac{\text{arm of } P}{\text{arm of } W}.$$

that is, the Power and the Weight are inversely proportional to the lengths of their respective arms.

100. **Mechanical Advantage of the Lever.**—It is evident that in a lever of the First Order, the power arm x may or may not be greater than the resistance arm y (fig. 55), and hence, there is a Mechanical Advantage or Disadvantage according as x is greater or less than y.

101. In a lever of the Second Order, x is always greater than y (fig. 56), hence W is always greater than P, and therefore a lever of the second order always acts at a Mechanical Advantage.

102. In a lever of the Third Order (fig. 57), x is always less than y; hence W is always less than P, and therefore a lever of the third order, such as the forearm, always acts at a Mechanical Disadvantage, and is, on this account, of no use in enabling us to overcome a large resistance by means of a small effort. If we remember, however, that, not alone in the case of a lever, but also in that of all the other mechanical powers where *power is gained, velocity is lost* and

vice versa, where *power is lost velocity is gained*, we can see that levers of the third order are of great use under circumstances where rapidity of motion is of more importance than an increase in force, for in this class of lever the loss of power is accompanied by a corresponding increase in the velocity. For instance, the limbs of animals, where rapidity of movement is of the utmost importance, are generally levers of the third order, or, when they are levers of the first order, they are so arranged that they act at a mechanical disadvantage—an example of the admirable adaptation of animal mechanism to the ends which are most fit and desirable.

Levers of the third order are also useful, as in the case of the tongs or forceps, for the application of force to bodies under circumstances where a direct application would not be convenient or desirable.

103. We have, in this account of the lever, assumed the bar itself to be straight and to have no weight; also the forces were supposed to act at right angles to its length.

In the case of bent levers, and also in cases where the weight of the lever is taken into account, and where the forces act obliquely, the conditions of equilibrium are obtained by equating the algebraic sum of the moments of the forces about the fulcrum to zero.

104. **Bent Levers.**—Let P and W (represented in fig. 58) be the power and weight respectively acting on a weightless

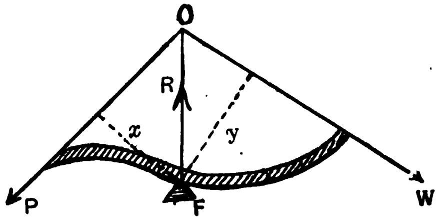

FIG. 58.

bent lever movable about a fulcrum F: produce the lines of action of P and W out to meet in the point O, and join OF.

It follows (Art. 83) that the line of action of the reaction R of the fulcrum at F must pass through O. Hence, if x and y represent the lengths of the perpendiculars from F on the lines of action of P and W respectively, we have, as before,

$$P \times x = W \times y, \quad \text{or} \quad \frac{W}{P} = \frac{x}{y}.$$

The value of R may be found by the Parallelogram of Forces, or by any of the methods adopted in Chapter III.

Similarly we may obtain the conditions of equilibrium of a bent lever of the second or third order.

Ex. 1. A weight of 10 lbs. balances a weight of 12 lbs. at the extremities of a weightless lever 11 inches long: find the lengths of the arms.

Here, let x = distance of F from the 10 lbs. (fig. 55);

$\therefore\ 11 - x$ = distance of F from the 12 lbs.

Take moments about F:

$\therefore\ 10x = 12(11 - x)$, or $10x = 132 - 12x$.

$\therefore\ 22x = 132$; $\therefore\ x = 6$ inches.

Hence the arms are of length 6 inches and 5 inches.

Ex. 2. If the pressure on the fulcrum of a lever be 3 lbs. wt., and the sum of the power and the weight 11 lbs. wt., find the whole length of the lever, given that the points of application of the power and weight are 6 inches apart.

The lever in this instance could not be a lever of the first order, as in that case the pressure on the fulcrum would be equal to the sum of the power and the weight (these being *like* parallel forces). Hence it must be of the second or third order, according as P is less than or greater than W. As it is immaterial which order we choose, let it be of the second order. The pressure on the fulcrum is, in this case, equal to the difference between the weight and the power.

Hence $W - P = 3$

and $W + P = 11$

$\therefore\ 2W = 14$;

$\therefore\ W = 7$ lbs. wt., and $P = 4$ lbs. wt.

Again, if $AF = x$ (fig. 56), $\therefore\ BF = x - 6$, we have, taking moments about F, $Px = W(x - 6)$, or $4x = 7(x - 6)$ $\therefore\ 3x = 42$, or $x = 14$ inches.

Ex. 3. Two forces P and W act at the ends A and B of a straight lever AB, and inclined to it at an angle of 45° and 60° respectively. The lever balances at a point O such that $AO = 9$ inches, and $BO = 6$ inches. Find the ratio of the forces to one another.

Here

$$P \times 9 \sin 45 = W \times 6 \sin 60, \text{ or } \frac{P \times 9}{\sqrt{2}} = \frac{W \times 6\sqrt{3}}{2}, \text{ or } \frac{P}{W} = \frac{\sqrt{6}}{3}.$$

EXAMPLES XII.

1. Two forces of 3 and 4 lbs. wt. act perpendicularly at the extremities of a straight lever 12 inches long: find the position of the fulcrum, the weight of the lever being neglected.

2. Two weights of 12 and 25 lbs. balance at the ends of a heavy lever which is divided by the fulcrum in the ratio of 2 : 1 : find the weight of the lever.

3. If the pressure on the fulcrum of a lever be 8 lbs. wt., and the sum of the power and weight 14 lbs. wt., find the whole length of the lever, given that the points of application of the forces are 4 feet apart.

4. The pressure on the fulcrum of a lever is 16 lbs. wt., and the difference of the forces is 2 lbs. wt.: find the ratio of the forces.

5. If the pressure on the fulcrum, in a lever of the first order, be 48 lbs. wt., and the arms are as 7 : 9, find P and W.

6. Find the length of a lever of the second order where a power of 10 lbs. wt. supports a weight of 30 lbs., their points of application being one foot apart.

7. The arms of a pair of nutcrackers are 6 inches long, and the nut is placed $\frac{3}{4}$ inch from the hinge. Assuming that a weight of 40 lbs. placed on the top of the nut would just crack it, find what force applied to the extremity of the arm will just break the nut.

8. Two forces of 3 and 4 units of weight act at the ends of a straight lever 12 inches long, and are inclined to the lever at angles of 30° and 45° respectively: find the position of the fulcrum when there is equilibrium.

9. If two weights balance, about a fixed fulcrum, at the extremities of a straight lever in any position inclined to the vertical, prove that they will balance about the same point in any other position.

10. A cubical block of stone, whose weight is 400 lbs., rests on the ground. A man thrusts one end of a lever, 3 feet long, to a distance of 6 inches under the block: find what upward force he must apply at the other extremity of the lever in order to just raise the block.

THE COMMON BALANCE.

105. The Common Balance is a lever of the First Order. It is used for testing whether two bodies have equal weights, and therefore equal masses (for the weights of bodies are proportional to their masses).

It consists of a rigid beam AB movable about a fulcrum at F, placed slightly above the centre of gravity of the beam.

The fulcrum at *F* consists of a wedge of hard steel whose edge, which is turned downwards, rests on a plate of agate.

The beam is thus supported by the upward pressure of the agate plate on the sharp edge of steel called the *knife-edge.*

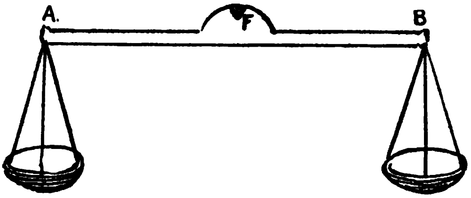

FIG. 59.

Two *scale-pans* are suspended, also by knife-edge supports, at *A* and *B.*

The body to be weighed is placed in one of the scale-pans, and weights are placed in the other pan until the beam remains in a horizontal position; then the sum of these weights will equal the weight of the body.

106. **Requisites of a good Balance.**—A good balance should be (1) *True,* (2) *Sensitive* (*or Sensible*), (3) *Stable.*

107. **True.**—A balance is said to be *true* if the beam remains horizontal when equal weights are placed in the scale-pans. This will be the case (*a*) if the weights of the scale-pans are equal, (*b*) if the arms are of equal length—that is, the fulcrum *F* must lie on a line bisecting *AB* at right angles, (*c*) if the fulcrum *F* and the centre of gravity of the beam are in the same vertical line when the beam is horizontal.

To test the *truth* of a balance, place a weight in one scale-pan and then place sufficient weight in the other scale-pan to keep the beam horizontal; then interchange the weights in the two pans, and, if there is no deflexion of the beam, the balance is true.

108. **Sensitive.**—A balance is said to be *sensitive* (or *sensible*) when a very small difference in the weights placed in the scale-pans produces an appreciable deflexion of the beam. That this may be the case (*a*) the arms must be as long as possible, (*b*) the beam should be as light as possible, (*c*) the fulcrum should be as small a distance as possible above the centre of gravity of the beam.

109. For, let A and B be the points of suspension of the scale-pans, and F the fulcrum, A, B, and F, as is usually the case, being in the same straight line (fig. 60). Let the weight of the beam be W, acting through G, the centre of gravity of the beam, the line GF being evidently at right angles to AB. Let the weights suspended at A and B be P and $P + w$ respectively, and let θ be the angle of deflexion of the beam from the horizontal line MN drawn through F. Also let the vertical through G cut MN in K.

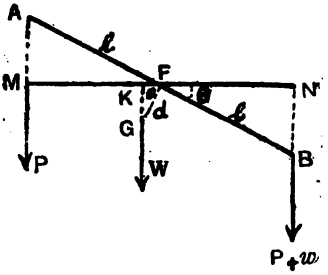

Fig. 60.

Let $AF = BF = l$, and $GF = d$.

Now, taking moments round F, we have

$$P \times FM + W \times FK = (P + w)FN;$$

but $FM = FN = l\cos\theta$, and $FK = d\sin FGK = d\sin\theta$:

$$\therefore \quad Pl\cos\theta + Wd\sin\theta = (P + w)\,l\cos\theta; \quad \therefore \quad Wd\sin\theta = wl\cos\theta;$$

$$\therefore \qquad \tan\theta = \frac{wl}{Wd}.$$

Hence, the greater the value of l, and the smaller the values of W and d, the greater will be the value of θ, and therefore the greater will be the *sensibility* of the balance.

109. **Stable.**—A balance is said to be *stable* when, if displaced, it returns to its position of equilibrium. That this may be the case, the centre of gravity of the beam should be below the point of suspension, and the greater the distance between these two points the more stable is the balance.

Note.—It follows from what we have just stated that, by increasing the stability of a balance, we make it less sensitive, and *vice versa.* In commerce, where rapidity of action is of importance, stability is most essential, whereas, in the case of the very delicate balances used in physical research, sensibility is of chief importance. Both sensibility and stability, however, can be ensured by making the arms long and light, and, at the same time, placing the fulcrum considerably above the centre of gravity of the beam.

110. **False Balance.**—A balance will not weigh accurately if the arms are unequal in length, even if the beam is horizontal when the scale-pans are not loaded; the true weight of a body, however, can be found by this form of false balance by either of the following methods of *double weighing*:—

111. **First Method.**—Let the lengths of the arms of the balance be a and b, and let the body, whose true weight is W, be balanced by a weight W_1 when placed at the extremity of the arm a, and by a weight W_2 when placed at the extremity of b.

Hence, we have $Wa = W_1 b$ and $Wb = W_2 a$.

On multiplying these results we obtain

$$W^2 ab = W_1 W_2 ab;$$

$$\therefore \quad W^2 = W_1 W_2 \quad \text{or} \quad W = \sqrt{W_1 W_2};$$

i.e. *the true weight is a geometric mean between the false weights.*

112. **Second Method.**—Place the body in one scale-pan, and balance it with sand, or any similar material placed in the other pan. Then remove the body and replace it with known weights sufficient to again balance the sand. The sum of these known weights will evidently be the weight of the body.

Ex. 1. A body when placed in one scale-pan of a false balance, whose arms are of unequal length, appears to weigh 9 lbs., and when placed in the other pan 7½ lbs.: find its true weight, and the ratio of the lengths of the arms.

Here, if a and b be the lengths of the arms (Art. 111),

$$Wa = 9b$$

and

$$\underline{Wb = 7\tfrac{1}{9}a;}$$

$$\therefore \quad W^2 ab = 64ab, \quad \text{or} \quad W^2 = 64, \quad \text{or} \quad W = 8 \text{ lbs. wt.}$$

Also the ratio of the arms is as 9 : 8.

Ex. 2. The arms of a false balance are as 8 : 7 ; and a shopman weighs alternately in each of the scale-pans so as to gain by one weighing and lose in the next, the same nominal weight of goods being in each case weighed out: show that he will be a loser at the end of an even number of sales to the extent of 1 lb. per cwt.

For, supposing 1 cwt. or 112 lbs. wt. of goods is asked for in each instance by the buyer, if the goods are placed at the extremity of the long arm, it is evident, by taking moments round the fulcrum, that the buyer only receives $\frac{7}{8}$. 112 lbs. or 98 lbs. This represents a gain to the shopman of 112 − 98 or 14 lbs.

Also, when goods are placed at the extremity of the short arm, the buyer receives $\frac{8}{7}$. 112 = 128 lbs. This represents a loss to the seller of 128 − 112 or 16 lbs. Hence, there is a net loss to the shopman, of 2 lbs. in a total sale of 2 cwt. of goods, or 1 lb. in each cwt. The same holds for any even number of sales.

The Common Steelyard.

113. The **Common** or **Roman Steelyard** consists of a bar *AB* movable about a fixed fulcrum at *F* (fig. 61).

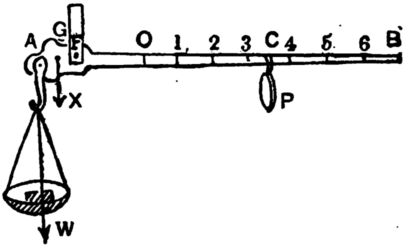

FIG. 61.

At *A* is suspended a scale-pan in which is placed the body to be weighed. The arm *FB* is graduated, and along this arm slides a movable weight *P*. The graduation at which *P* rests when it equilibrates the weight of the body in the scale-pan will give the weight of the body.

114. **To Graduate the Common Steelyard.**—Let *X* be the combined weight of the beam and scale-pan acting at *G*, the common centre of gravity of the beam and scale-pan (this point being generally situated in the arm *AF*).

The *zero* point 0 is first found thus :—

The movable weight P is shifted along the bar FB to a point such that the steelyard just balances when *no* weight is in the pan. Then, taking moments about F, we have

$$X \,.\, GF = P \,.\, OF.$$

Now, let a weight W be placed in the pan, and let C be the position of P when the steelyard again balances ; then

$$W \,.\, AF + X \,.\, GF = P \,.\, CF.$$

But $$X \,.\, GF = P \,.\, OF;$$

therefore subtracting

$$W \,.\, AF = P\,(CF - OF) = P \,.\, OC;$$

$$\therefore \quad W = \frac{OC}{AF} \,.\, P.$$

Now, if $OC = AF$, then $W = P$.

Also, if $OC = 2AF$, then $W = 2P$.

And, if $OC = 3AF$, then $W = 3P$. &c.

Hence, distances are measured off from 0 equal to AF, $2AF$, $3AF$, etc., and marked with the respective numbers 1, 2, 3, etc. ; these, when read off, give the number of times the weight of the body contains P lbs. For example, if the weight of P were 5 lbs., and P was on the 10th division when it balanced a body in the scale-pan, then the weight of the body would be 50 lbs.

The spaces between the numbers 0, 1, 2, 3, etc., are usually subdivided to indicate fractional parts of the movable weight P.

The Danish Steelyard.

115. This consists of a bar AB whose fulcrum F is movable; the bar terminates at B in a heavy knob, and at A is suspended a scale-pan (fig. 62).

Let X be the combined weight of the bar and scale-pan acting at G, their common centre of gravity, and let F be the point to which the fulcrum has to be moved in order that the steelyard should balance when a weight W is placed in the scale-pan. Hence, taking moments round F, we have

$$W \,.\, AF = X \,.\, GF = X(AG - AF);$$

$$\therefore \quad (W + X)AF = X \,.\, AG;$$

$$\therefore \qquad AF = \frac{X}{W + X} \,.\, AG.$$

Now, let $W = X$; $\quad \therefore \quad AF = \frac{1}{2} AG.$

Also, let $W = 2X$; $\quad \therefore \quad AF = \frac{1}{3} AG.$

And, let $W = 3X$; $\quad \therefore \quad AF = \frac{1}{4} AG$, etc.

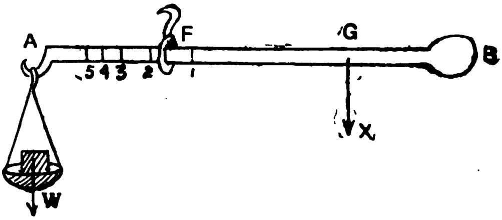

FIG. 62.

Hence, bisect AG and mark the point with the figure 1, and, when the fulcrum is at this point, the weight of the body is equal to X. Again, trisect AG and mark the point of trisection which is nearest A with the figure 2, and, when the bar balances at this point, the weight in the pan must be $2X$. Similarly it can be marked with the numbers 3, 4, 5, etc.

As before, the distances between the numbers 1, 2, 3, etc., can be subdivided to measure fractional parts of X.

EXAMPLES XIII.

1. The arms of a false balance are of unequal length, and a body placed in one scale-pan appears to weigh 8 lbs., while, if placed in the other pan, its apparent weight is $6\frac{1}{8}$ lbs.: find the true weight of the body and the ratio of the arms.

2. When placed in one pan of a false balance a body appears to weigh 15 lbs., and when placed in the opposite pan 16 lbs.: find the true weight of the body, assuming that the balance is false owing to the arms being of unequal length.

3. The arms of a balance are as 14 : 15, and a commodity is weighed as often in one pan as in the other: find the loss per cent. to the seller after an even number of sales.

4. The arms of a balance are of equal length, but the balance is false owing to one scale-pan being loaded: show that, if a body appears to weigh W_1 and W_2 when weighed in each scale-pan in succession, the true weight W is given by the equation

$$W = \tfrac{1}{2}(W_1 + W_2),$$

i.e. the true weight is the arithmetic mean of its apparent weights.

Note.—The student will notice that in this case there is neither a gain nor loss to the seller after an even number of sales.

5. If a balance be false owing to the two scale-pans being of unequal weights, and a body appears to weigh 14 lbs. when placed in one scale, and 14½ lbs. when weighed from the other end, find the true weight of the body.

6. The arms of a balance are of lengths 1 foot 3 inches and 1 foot 4 inches; when a body is suspended at the extremity of the shorter arm, it appears to weigh 7½ lbs.: find its true weight.

7. If the price of tobacco be 8 shillings per lb., and it is weighed in a false balance whose arms are in the ratio 15 : 16, find the actual price paid by the buyer when the tobacco is placed at the extremity (1) of the longer, (2) of the shorter, arm.

8. The common centre of gravity of a steelyard and the scale-pan is 5 inches from one end, and the fulcrum is 6 inches from the same end; the combined weight of yard and pan is 8 lbs., and the movable weight is 1½ lbs.: find the position of the zero-point, and the distance between the lb. graduations.

9. In a common steelyard the distance of the fulcrum from the point of support of the weight is 1 inch, and the movable weight is 6 ozs. In order to weigh 15 lbs., the movable weight must be placed 8 inches from the fulcrum: where must it be placed in order to weigh 20 lbs.?

10. A uniform bar 3 feet long weighing 6 lbs. is used as a steelyard, being supported at a point 5 inches from one end: find the greatest and least weights that can be weighed with a movable weight of 2 lbs.

11. In a Danish steelyard the zero is 10 inches from the end at which the body to be weighed is attached; the weight of the beam is ¾ lb.: find the position of the graduation corresponding to a weight of 20 ozs.

The Wheel and Axle.

116. The **Wheel and Axle** is a modification of the lever. It consists of two circular cylinders of unequal radii which can revolve round a common central axis. The larger CD is called the **wheel**, and the smaller AB the **axle** (fig. 63). The axle terminates in two pivots at X and Y, which rest on fixed supports, and around which it can turn freely. Ropes are wound round the wheel and axle in opposite directions. To the extremity of the rope round the axis is attached the weight W, and at the extremity of that round the wheel is applied the power P, which may either consist of another weight or a pull applied to the rope. The sockets on which the two pivots at X and Y rest act as fulcra; and the pressure they sustain is the resultant of P and W.

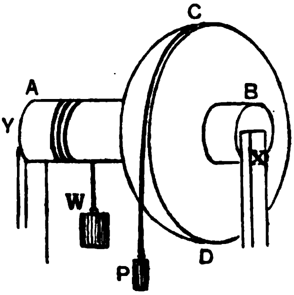

FIG. 63.

To find the Relation between the Power and the Weight in the Wheel and Axle.

117. Let fig. 64 represent a cross-section of the wheel and axle by a vertical plane where the wheel meets the axle; and we may suppose both P and W to act in the plane of this section, the whole of this machine constituting a lever of the first order, the fulcrum F being the centre of the axle. If the radii of the wheel and axle be denoted by R and r respectively, we have, since the moments of P and W about the fulcrum are equal and opposite,

$$P \times R = W \times r,$$

or P × radius of wheel = W × radius of axle.

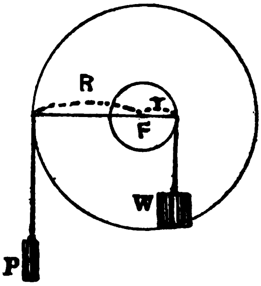

FIG. 64.

118. From this we can find the Mechanical Advantage of the wheel and axle, for

$$\textbf{Mechanical Advantage} = \frac{W}{P} = \frac{R}{r} = \frac{\textbf{radius of wheel}}{\textbf{radius of axle}}.$$

119. The mechanical advantage can therefore be made very great by making the radius of the wheel very large and that of the axle very small; there is, however, a limit to this increase or decrease; for too large a wheel would be cumbersome, and an axle of very small radius would lack sufficient strength to sustain heavy loads.

120. The **Windlass.**—The most common example of the wheel and axle is the *Windlass.* The Power is applied by means of a winch-handle or spokes, the circle described by the winch-handle or by the extremities of the spokes constituting the wheel, and the horizontal cylinder round which the rope, which carries the Weight, is coiled being the axle. It is often seen in use for raising buckets of water from a well, or portions of the cargo from the hold of a ship.

121. The **Capstan.**—Another form of the wheel and axle is the *Capstan,* which differs only from the windlass in that the axle is vertical instead of being horizontal. It is used on ships for raising the anchor.

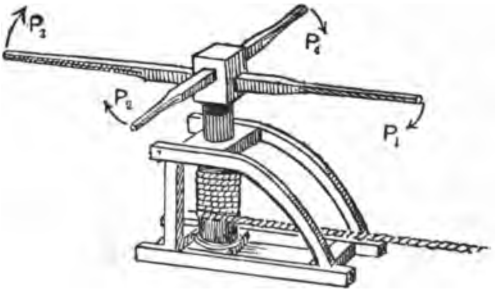

Fig. 65.

The Power consists of forces P_1, P_2, etc. (fig. 65), applied at the extremities of bars called handspikes inserted into sockets in the axle, the Weight being the tension of the rope coiled round the axle.

Ex. 1. If the radius of the wheel be 2 feet, and that of the axle 3 inches, find what power would be sufficient to raise a weight of 20 lbs.

Here $P \times$ radius of wheel $= W \times$ radius of axle.

$$P \times 24 = 20 \times 3, \quad \therefore \quad P = 2\tfrac{1}{2} \text{ lbs. wt.}$$

Ex. 2. If the radius of the axle be 8 inches, and that of the wheel 4 feet, find what weight will be sustained by a power of 20 lbs. wt., the rope which is coiled round the axle being 1 inch thick.

In this case half the thickness of the rope must be added to the radius of the axle, which will now be $8\frac{1}{2}$ inches.

$$\therefore \quad 20 \times 48 = W \times 8\tfrac{1}{2}; \quad \therefore \quad W = \frac{20 \times 48 \times 2}{17} = 112\tfrac{16}{17} \text{lbs.}$$

122. **The Differential Wheel and Axle.**—In this arrangement the axle is not of the same thickness throughout, as it consists of two cylinders of different radii, which, when in motion, revolve round a common central axis round which the wheel also revolves. A section, similar to that given for the common wheel and axle, is seen in the accompanying diagram (fig. 66).

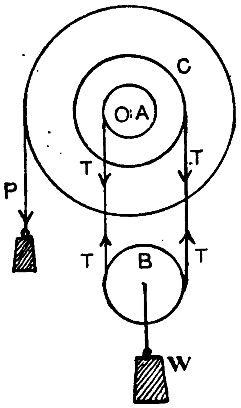

FIG. 66.

The rope is wound round the small cylinder A, and after passing round a pulley B is wound in the contrary direction round the large cylinder C. The weight W is suspended from the pulley, and the power P is applied, as in the ordinary wheel and axle, to the circumference of the wheel.

Let the radii of the large and small cylinders be r and r' respectively, and that of the wheel R. Also, let the tension in each part of the string which supports the weight be T (supposed parallel in direction). Hence, for equilibrium $2T = W$. Also, taking moments about O and equating the clockwise to the contraclockwise rotations, we have $P.R + T.r' = Tr$, or $PR = T(r - r')$;

but $T = \frac{W}{2}$; $\therefore$ $P.R = \frac{W}{2}(r - r')$.

And the Mechanical Advantage

$$\frac{W}{P} = \frac{2R}{r - r'}.$$

123. The advantage of this form of the wheel and axle is that, by making r and r' very nearly equal, and hence $r - r'$ very small, we can make the ratio of the weight to the power as large as we choose.

Ex. If the radii of the two cylinders are 5 inches and 3 inches respectively, and that of the wheel 2 feet, find what weight will be lifted by a power of 15 lbs. wt.

Here $15 \times 24 = \frac{W}{2}(5 - 3)$; $\therefore$ $W = 360$ lbs.

EXAMPLES XIV.

1. Find the radius of the wheel which will enable a power of 2 lbs wt. to support a weight of 28 lbs., the diameter of the axle being 6 inches.

2. Find what weight suspended from the axle can be supported by a weight of 3 lbs. suspended from the wheel, if the radius of the axle is 2 feet, and that of the wheel 9 feet. Find also the total pressure on the supports on which the axle rests.

3. Show how to construct a wheel and axle which will enable a power of 28 lbs. wt. to support a weight of 3 cwts.

4. Show how to construct a wheel and axle so as to give a mechanical advantage of $5\frac{1}{2}$.

5. If a power of 5 lbs. wt. balances a weight of 45 lbs., and if the radius of the wheel be 2 feet, find the radius of the axle. Also what is the mechanical advantage?

6. If the diameter of the wheel be 5 feet and the radius of the axle 6 inches, find what weight could be sustained by a power of 20 lbs. wt., both the power and weight being applied at the extremities of ropes each $\frac{1}{2}$ an inch thick.

7. Six sailors, each exerting a force equal to a weight of 112 lbs., raise an anchor by means of a capstan, the levers of which are 8 feet long, the radius of the axle being 18 inches: find the weight of the anchor, assuming that 25 per cent. of the power is lost in overcoming friction.

8. Three wheels and axles are arranged so that by means of teeth the circumference of each axle fitted to that of the next wheel: if the radius of each wheel be 1 foot and of each axle 3 inches, find what power, applied at the circumference of the first wheel, will support a weight of 320 lbs. hanging from the axle of the last.

9. If the mechanical advantage of a wheel and axle be 8, and the difference of the radii of the wheel and axle $3\frac{1}{2}$ feet, find the radii.

10. If the circumferences of the wheel and axle be $7\frac{1}{2}$ feet and 10 inches respectively, find what power would raise a weight of $4\frac{1}{2}$ cwt.

11. In the differential wheel and axle, if the axle consists of two cylinders of radii $2\frac{1}{2}$ inches and 2 inches respectively, the radius of the wheel being 1 foot, find what power will support a weight of 480 lbs.

12. The axle of a capstan is 20 inches in diameter and there are 6 bars. At what distance from the axis must 6 men push, one at each bar and each exerting a force equal to a weight of 24 lbs., in order to just raise an anchor of weight 1 ton?

13. In the differential wheel and axle, if the axle consists of two cylinders of radii 4 and 3 inches respectively, the radius of the wheel being 2 feet, find what weight will be lifted by a power of 20 lbs. wt.

14. If the mechanical advantage of a wheel and axle be 11 and the sum of the radii of the wheel and axle be 3 feet, find the radii.

CHAPTER IX.

THE PULLEY AND INCLINED PLANE.

124. **Pulleys.**—A *pulley* consists of a small wheel which can turn freely round an axis perpendicular to its plane; the circumference is usually grooved in order to prevent the cord, which passes over the pulley, from slipping off. The axis round which the wheel can revolve is supported by a framework of wood or metal called the *block* (fig. 67). In fact, the pulley may be regarded as a lever of the first order with equal arms; hence the power P at one end of the string (fig. 67) must equal the weight W at the other end; and this will hold whether P and W are parallel to one another or inclined at any angle. Hence the principle on which the equilibrium of a system of pulleys is determined is the following:—

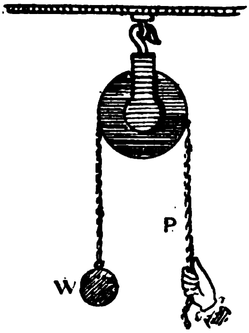

Fig. 67.

The Tension of each string in the system is the same throughout its length, and is equal to the Power applied at its extremity.

125. **The Single Movable Pulley.**—The weight W is attached to the block which contains the movable pulley A; the power P is applied at the extremity of a cord which passes round a fixed pulley B which is attached to a beam above at C; the cord then passes round the movable pulley, and its other extremity is tied to the beam at D (fig. 68).

In this arrangement, supposing the two portions of the string on either side of A to be parallel to one another, then the tension of each portion being equal to P the power applied at the extremity, the weight W acting vertically downwards is kept in equilibrium by two parallel forces each equal to P acting vertically upwards; hence $W = 2P$.

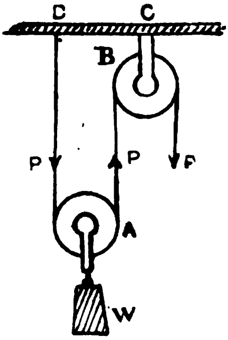

FIG. 68.

Also, if the weight of the movable pulley, including the block, be w, we have

$$W + w = 2P.$$

126. Again, let the portions of the string round A be inclined at an angle 2θ (fig. 69). Then, since the tensions of the two portions of the string are equal, the line of action of W must bisect the angle between them; and, resolving the forces vertically, we have for equilibrium:—

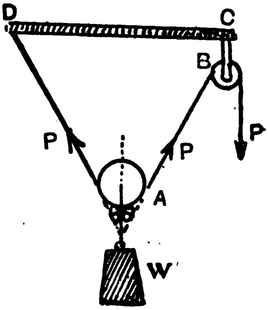

FIG. 69.

$$W = P\cos\theta + P\cos\theta = 2P\cos\theta,$$

or, taking into account the weight w of the pulley, $W + w = 2P\cos\theta$.

Note.—The same results could be obtained in both these cases without the aid of the pulley B by pulling the string in the direction AB with a force equal to P, the only use of the fixed pulley being to enable the power to be conveniently applied in a downward direction.

127. **First System of Pulleys.**—*In which each movable pulley hangs by a separate string, one end of which is fastened to a beam above, all the strings being parallel.* Let A_1, A_2, A_3, be the movable pulleys. The power P may either be applied downwards, the string passing round a fixed pulley at B, or may be applied upwards along the direction A_3B, in which

case the fixed pulley is not required. The string at the extremity of which P is applied passes under the movable pulley A_3 and is fastened to the beam above at F. Another string fastened to the block of A_3 passes under the pulley A_2, and is fastened above at G, and similarly for the other strings. The weight W is hung from the lowest pulley.

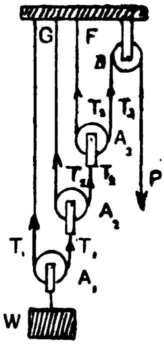

Fig. 70.

128. **Case I.**—Where the weights of the pulleys are neglected. Let the tensions of the strings, commencing with the lowest, be T_1, T'_2, T_3, &c., respectively. Now the pulley A_1 is in equilibrium under the action of the parallel forces T_1 and T_1 upwards and W downwards;

$$\therefore\ 2T_1 = W;\quad \therefore\quad T_1 = \frac{W}{2}.$$

Again, considering the equilibrium of the pulley A_2, we have

$$2T'_2 = T'_1;\quad \therefore\quad T_2 = \frac{T_1}{2} = \frac{W}{2^2}.$$

Similarly $T_3 = \frac{W}{2^3}$, and $P = T_3 = \frac{W}{2^3}$,

where there are three movable pulleys. And when there are n movable pulleys

$$P = \frac{W}{2^n} \quad\text{or}\quad W = 2^n P.$$

Hence the *mechanical advantage* where the weights of the pulleys are neglected $= \frac{W}{P} = 2^n$.

129. **Case II.**—Let the weights of the pulleys counting from below upwards be $w_1, w_2, w_3, \dots w_n$ respectively. Since, to support a weight W, a power of $\frac{W}{2^n}$ is required, therefore, if the weight of the pulley A_1 be w_1, this power has to be increased by $\frac{w_1}{2^n}$.

Similarly, if the weight of A_2 be w_2, the power must be increased by $\frac{w_2}{2^{n-1}}$, and, for the pulley A_3, it must be increased by $\frac{w_3}{2^{n-2}}$, &c. Therefore, if there are n movable pulleys, we have

$$P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \ldots + \frac{w_n}{2}.$$

130. **Total Stress on the Beam.**—Let R be the resultant stress on the beam. Now it is evident that, where P is applied vertically downwards (a fixed pulley being used), the stress R is equal to the sum of W, the weights of the separate pulleys including the fixed pulley, and P; and where P is applied vertically upwards R is equal to the sum of W and the weights of the pulleys minus P.

N.B.—It is advisable for the student not to make any use of the above formula in numerical examples, but to apply the principles involved in the proof of the formula to each separate case.

Ex. 1. Draw a diagram of a system of pulleys in which, if the weights of the pulleys are not taken into account, the mechanical advantage will be 16. (2) Supposing each pulley to weigh 1 lb., find what power will sustain a weight of 145 lbs.

Here $16 = 2^4$; hence there are 4 separate strings.

Again, let the tensions from below upwards be T_1, T_2, T_3, and T_4 respectively; and we have

$$2T_1 = 145 + 1, \quad \therefore T_1 = 73.$$
$$2T_2 = 73 + 1, \quad \therefore T_2 = 37.$$
$$2T_3 = 37 + 1, \quad \therefore T_3 = 19.$$
$$2T_4 = 19 + 1, \quad \therefore T_4 = P = 10 \text{ lbs. wt.}$$

Ex. 2. If there be three movable pulleys in which each pulley hangs by a separate string, one end of which is fastened to a beam above, the weights of the pulleys, commencing with the lowest, being 3, 2, and 1 lbs. respectively, find what weight a power of 5 lbs. wt. will support. Find also the resultant stress on the beam. In this case the diagram is the same as in fig. 70.

Now, commencing with P, we have

$$T_3 = P = 5;$$

also

$$T_2 + 1 = 2T_3 = 10, \quad \therefore T_2 = 9.$$
$$T_1 + 2 = 2T_2 = 18, \quad \therefore T_1 = 16.$$
$$W + 3 = 2T_1 = 32, \quad \therefore W = 29 \text{ lbs.}$$

Also, if R = the resultant stress on the beam, the power being supposed to be applied vertically upwards,

$$R = 29 + 3 + 2 + 1 - 5 = 30 \text{ lbs. wt.}$$

Ex. 3. A man of 170 lbs. wt., sitting in a loop or seat, is suspended from the lowest of a system of 4 weightless pulleys in which each pulley hangs by a separate string, and supports himself by pulling at the end of the string which passes round the fixed pulley. Find the force with which he pulls the string, the weight of the seat being neglected.

Here the force P with which he pulls the string actually supports part of his weight, and if W represent the weight of the man, the resistance overcome is $W - P$, $\therefore$ as before, we have

$$W - P = 16P,$$

$$170 - P = 16P,$$

or

$$17P = 170 : \quad \therefore P = 10 \text{ lbs. wt.}$$

EXAMPLES XV.

1. In a single movable weightless pulley with parallel strings, find what force is necessary to support a weight of 100 lbs. Also what is the mechanical advantage?

2. If in Ex. 1 the movable pulley weighed 8 lbs., find what force would be necessary to support the 100 lbs. wt.

3. In a single movable pulley whose weight is 6 lbs., find what weight a power of 12 lbs. wt. could sustain.

4. In the first system of pulleys, if there are 5 movable weightless pulleys, find what power will support a weight of 160 lbs.

5. In the first system of pulleys, if there are 6 movable weightless pulleys, find what power will support a weight of 1 ton.

6. In the first system of pulleys, if there are 4 movable weightless pulleys, find what weight a power of 2 lbs. wt. will support: also, if the direction of the power be vertically upwards, find the stress on the fixed beam.

7. In the first system of pulleys, if a power of 7 lbs. wt. sustains a weight of 56 lbs., find the number of pulleys, the pulleys being weightless.

8. In a system of weightless pulleys where each pulley hangs by a separate string, the extremity of each being tied to a beam above, if the mechanical advantage be 128, find the number of movable pulleys.

9. Draw a diagram of a system of pulleys in which, if no account be taken of the weights of the pulleys, the mechanical advantage shall be 8. (*a*) If, in this system, each of the pulleys weighs 2 lbs., find (1) what force will support a weight of 34 lbs.; (2) through what distance must the point of application of P move in order that W may be raised 10 inches?

10. Draw a diagram of a system of pulleys in which, the pulleys being regarded as weightless, the weight shall be to the power as 16 : 1. Also if, in the same system, the weights of the movable pulleys from below upwards are 4, 3, 2, and 1 lbs. respectively, find what force will support a weight of 102 lbs.

11. If, in the first system, there are 3 movable pulleys, whose weights, commencing with the lowest, are 4, 3, and 2 ozs. respectively, find what force will sustain a weight of 18 ozs. Also if the point of application of P move vertically downwards by 2 feet, through what distance will W be raised?

12. In the first system of pulleys where there are 5 movable pulleys each weighing 8 ozs., find what weight a force of 5 lbs. wt. will sustain.

13. Draw a diagram of a system in which, without taking into account the weights of the pulleys, the mechanical advantage shall be 32. Also, in the same system, if each pulley weighs 12 ozs., find what weight a power of 6 lbs. wt. will sustain.

14. In the first system of pulleys where there are 4 movable pulleys all of equal weight, it is found that a force of 5 lbs. wt. can sustain a weight of 50 lbs.: find the weight of each pulley.

15. If, in the first system, a force of 12 lbs. wt. sustains a weight of 85½ lbs., each pulley weighing 1½ lbs., find the number of movable pulleys.

16. A man whose weight is 12 stone, seated at one end of a loop of a rope which passes over a single *fixed* pulley, supports himself by pulling at the other end: find what force he exerts.

17. A man whose weight is 12 stone, sitting in a loop or seat suspended from a single weightless movable pulley, supports himself by pulling at the end of the rope, which passes over a fixed pulley: find the force he exerts, the weight of the loop or seat being neglected, and all the strings being supposed to be vertical.

18. In the last example, supposing the seat on which the man sits to be suspended from the lowest of 3 movable pulleys, each of which weighs 3 lbs., find what force he must exert.

131. **Second System of Pulleys.**—*Where the same string passes round all the pulleys.* In this system there are two blocks, each containing several pulleys. The upper block is fixed to the beam above, while the lower block, to which the weight is suspended, is movable.

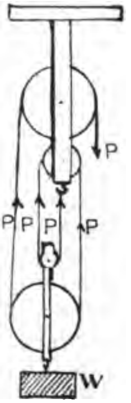

FIG. 71.

If the number of portions of string connecting the two blocks be *even* (as in fig. 71, in which there are four portions), which will be the case when there is the same number of pulleys in the upper as in the lower block, the end of the string will be attached to a hook in the upper block (fig. 71). If, however, the number of portions of string be *odd*, which will happen when there is one more pulley in the upper than in the lower block, the end of the string will be attached to a hook in the lower block. In either case, if a power P be applied at the extremity of the string, then the tension of each portion of the string is equal to P, and if there are n portions of string, supposed parallel to one another, the weight W is equilibrated by a total upward force of nP; we have, therefore, if we neglect the weight of the lower block,

$$W = nP, \text{ the mechanical advantage being } \frac{W}{P} \text{ or } n.$$

Again, if w is the weight of the lower block, we have

$$W + w = nP.$$

Ex. 1. If there are 3 pulleys in each block, find what force would sustain a weight of 144 lbs., neglecting the weight of the lower block.

There will, therefore, be 6 strings supporting the weight.

$$\therefore\ W = 6P \quad \text{or} \quad 144 = 6P.$$

$$\therefore\ P = 24 \text{ lbs. wt.}$$

Ex. 2. If a power of 5 lbs. wt. supports a weight of 18 lbs., and a power of 7 lbs. wt. supports 26 lbs., find the number of strings, and the weight of the lower block.

Here $\quad W + w = nP;$

$\therefore \quad 18 + w = 5n;$

also $\quad$ [illegible]6 $+ w = 7n.$

$\therefore \quad -8 = -2n$

$\therefore \quad n = 4, \quad \text{and} \quad w = 2 \text{ lbs.}$

EXAMPLES XVI.

1. Find the number of strings at the lower block, and the number of pulleys in each block, when a power of 2 ozs. wt. supports a weight of 1 lb., neglecting the weight of the lower block.

2. If a power of 4 lbs. wt. supports a weight of 18 lbs., the weight of the lower block being 2 lbs., find the number of strings.

3. When $W = 25$, $P = 5$; and when $W = 49$, $P = 8$; find the number of strings, and the weight of the lower block.

4. If, in the second system of pulleys, a man standing on the ground, whose weight is 12 stone, supports a weight of 5 cwt., the number of strings being 5: neglecting the weight of the lower block, find his pressure on the ground, supposing all the strings to be vertical.

5. In the last example, supposing the weight of the lower block were 20 lbs., find what would then be the man's pressure on the ground.

6. A man, weighing 12 stone, supports himself in a basket which is suspended from the lower block in the second system of pulleys, where there are 3 strings attached to the lower block: neglecting the weight of the basket and that of the lower block, find what force he exerts, assuming that he pulls vertically.

7. If, in the last example, the weight of the man be W, and each block to contain 4 pulleys, find what vertical force he exerts.

132. **The Third System of Pulleys.**—*Where each string is attached to the weight.*

In this system each movable pulley hangs by a separate string, which passes over the pulley above it, and has its other end fastened to a bar from which the weight is suspended.

In fig. 72 is represented a system of this kind where there are 4 strings.

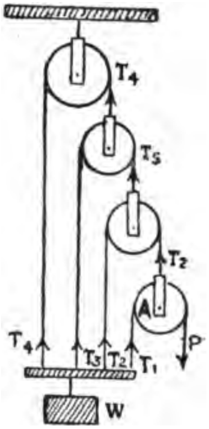

FIG. 72.

133. **Case I.**—Where the weights of the pulleys are neglected.

Let the tensions of the strings, commencing with the lowest, be T_1, T_2, T_3, &c., respectively (fig. 72). Now, the pulley A is in equilibrium under the action of T_2 upwards, and $P + T_1$ or $2T_1$ downwards;

$$\therefore \quad T_2 = 2T_1 = 2P.$$

Similarly $$T_3 = 2T_2 = 2^2P,$$

And $$T_4 = 2T_3 = 2^3P.$$

But, since all the strings are attached to the weight, W must equal the sum of the tensions of the strings, or

$$\begin{aligned} W &= T_4 + T_3 + T_2 + T_1 \\ &= 2^3P + 2^2P + 2P + P \\ &= (2^3 + 2^2 + 2 + 1)P = \frac{2^4 - 1^*}{2 - 1}P, \end{aligned}$$

or $W = (2^4 - 1)P$, where there are 4 strings.

Similarly $W = (2^5 - 1)P$, where there are 5 strings, etc.,

and $\therefore$ $W = (2^n - 1)P$, for n strings;

and the mechanical advantage

$$\frac{W}{P} = 2^n - 1.$$

---

* This is evident from the fact that $x^3 + x^2 + x + 1 = \frac{x^4 - 1}{x - 1}$.

134. **Case II.**—Let the weights of the movable pulleys, counting from below upwards, be w_1, w_2, w_3, etc.

Then, considering the equilibrium of the pulley A (fig. 72), we have, equating the upward force to the sum of the forces downwards,

$$T_2 = 2T_1 + w_1 = 2P + w_1.$$

Similarly
$$T_3 = 2T_2 + w_2 = 2^2P + 2w_1 + w_2$$
$$T_4 = 2T_3 + w_3 = 2^3P + 2^2w_1 + 2w_2 + w_3.$$

But, since all the strings are attached to the weight, W must equal the sum of the tensions of the strings, or

$$\begin{aligned} W &= T_4 + T_3 + T_2 + T_1 \\ &= (2^3 + 2^2 + 2 + 1)\,P + (2^2 + 2 + 1)\,w_1 + (2 + 1)\,w_2 + w_3 \\ &= \frac{2^4 - 1}{2 - 1}\,P + \frac{2^3 - 1}{2 - 1}\,w_1 + \frac{2^2 - 1}{2 - 1}\,w_2 + w_3, \end{aligned}$$

or
$$W = (2^4 - 1)\,P + (2^3 - 1)\,w_1 + (2^2 - 1)\,w_2 + w_3,$$

where there are 4 pulleys (including the fixed one whose weight need not be considered, as it is supported by the beam), and therefore 4 strings.

Similarly, if there are n pulleys, including the fixed pulleys, and therefore n strings, we have

$$W = (2^n - 1)\,P + (2^{n-1} - 1)\,w_1 + (2^{n-2} - 1)\,w_2 + \ldots$$
$$+ (2^2 - 1)\,w_{n-2} + (2 - 1)\,w_{n-1}.$$

N.B.—The student is advised not to make use of the above formulæ in the case of numerical examples, but to work out such examples from first principles.

135. In the Third System the weights of the pulleys employed assist the Power, and, therefore, they should be made as heavy as possible, whereas in the First and Second Systems their weights act against the Power, and the pulleys should, under these circumstances, be made as light as possible consistent with strength.

Ex. 1. Draw a diagram of a system of pulleys in which, if the weights of the pulleys are not taken into account, the power will be to the weight as 1 to 31; (2) supposing each pulley to weigh 1 lb., find what weight a power of 10 lbs. wt. will sustain.

Here $31 = 32 - 1 = 2^5 - 1$; and we conclude that the third system of pulleys with 5 strings will have the required mechanical advantage. Again, if each pulley weigh 1 lb., we have:—

$$\begin{aligned} T_1 &= P = 10, \\ T_2 &= 2T_1 + 1 = 21, \\ T_3 &= 2T_2 + 1 = 43, \\ T_4 &= 2T_3 + 1 = 87, \\ T_5 &= 2T_4 + 1 = 175, \\ \therefore\quad W &= T_1 + T_2 + T_3 + T_4 + T_5 = 336 \text{ lbs.} \end{aligned}$$

Ex. 2. In the system of pulleys where each string is attached to the weight, where there are 4 pulleys, and the movable pulleys, commencing with the lowest, weigh 1, 2, and 3 lbs. respectively, find what power will support a weight of 91 lbs.

Let P be the power.

Then

$$\begin{aligned} T_1 &= P, \\ T_2 &= 2T_1 + 1 = 2P + 1, \\ T_3 &= 2T_2 + 2 = 4P + 4, \\ T_4 &= 2T_3 + 3 = 8P + 11, \\ \therefore\quad W &= T_1 + T_2 + T_3 + T_4 = 15P + 16; \end{aligned}$$

or

$$91 = 15P + 16, \quad \therefore\ P = 5 \text{ lbs. wt.}$$

Ex. 3. In the last example, if the bar (fig. 72) to which W is attached be of length a between the extreme strings, find at what point of the bar the weight W must be attached in order that the bar may remain horizontal, assuming that the points at which the strings are attached to the bar are at equal intervals apart.

Let x = distance of the point at which W is hung from the point of application of T_4. Then the bar is in equilibrium under the action of five parallel forces, viz. 91 downwards, and T_1, T_2, T_3, T_4 upwards, which are respectively equal to 5, 11, 24, and 51 lbs. wt. Then taking moments

$$91x = 51 \times 0 + 24 \times \frac{a}{3} + 11 \times \frac{2a}{3} + 5 \times a, \quad \therefore\ x = \frac{61a}{273}.$$

EXAMPLES XVII.

1. Draw a diagram of a system of pulleys in which, if the weights of the pulleys are not taken into account, the power shall be to the weight as 1 to 7: (a) supposing each pulley to weigh 2 lbs., find what weight a power of 10 lbs. wt. will sustain.

2. Draw a diagram of a system of pulleys in which, supposing the pulleys to be weightless, the mechanical advantage shall be 15. (a) Supposing each pulley to weigh 3 lbs., find what weight a power 10 lbs. wt. will sustain. (b) Through what distance must the point of application of P be moved in order that W may be raised 6 inches?

3. In the system of pulleys where each string is attached to the weight, if there are four pulleys, the weights of those which are movable, commencing with the lowest, being 4, 5, and 6 lbs. respectively, find what power will support a weight of 169 lbs.

4. If, in the system where each string is attached to the weight, the mechanical advantage, on the supposition that the pulleys are weightless, be 63, find what power would support a weight of 744 lbs., supposing the pulleys to each weigh 2 lbs.

5. In the third system of pulleys, if each pulley weighs 20 ounces, there being 3 movable pulleys, find what weight a power of 3 lbs. wt. can sustain.

6. In the third system three movable pulleys, each weighing 12 lbs., are used to sustain a weight 19 cwt. 1 qr. 16 lbs.: find the power.

7. If, in the last example, the power be applied by a man whose weight is 12 stone standing on the ground, and pulling vertically downwards, find the pressure of the man on the ground while he sustains the weight.

8. If there be 4 weightless pulleys, of which one is fixed, each of radius 3 inches, and the strings attached to a light bar which supports the weight, find at what point the weight should be attached in order that the bar should remain in a horizontal position.

136. Besides the three systems of pulleys just described, in which the usual order has been followed, we may notice a fourth system in which each cord, instead of being fastened to the beam above, is made to pass over a fixed pulley, and is then brought down and attached to the upper part of its own block (fig. 73). In this case, if we neglect the weight of the movable pulleys, we have

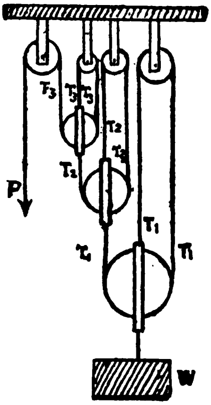

FIG. 73.

$$3T_1 = W; \quad \therefore \ T_1 = \frac{W}{3},$$

$$3T_2 = T_1; \quad \therefore \ T_2 = \frac{T_1}{3} = \frac{W}{3^2},$$

Similarly, $$T_3 = \frac{T_2}{3} = \frac{W}{3^3}; \quad \therefore \ P = T_3 = \frac{W}{3^3},$$

where there are three movable pulleys. And where there are n movable pulleys,

$$P = \frac{W}{3^n} \quad \text{or} \quad W = 3^n P.$$

And the mechanical advantage $= \dfrac{W}{P} = 3^n$.

Again, if the weights of the movable pulleys, counting from below upwards, be w_1, w_2, w_3, . . . w_n respectively, it may be proved, as in Art. 129, that

$$P = \frac{W}{3^n} + \frac{w_1}{3^n} + \frac{w_2}{3^{n-1}} + \ldots + \frac{w_n}{3}.$$

EXAMPLES XVIII.

1. Draw a diagram of a system of pulleys in which, if the pulleys are weightless, the mechanical advantage may be 27. (*a*) If each movable pulley weighs 2 lbs., find what weight a power of 10 lbs. wt. will support. Also find in the latter case the total stress on the supporting beam, the fixed pulleys being regarded as weightless.

2. In this system, if there are two movable pulleys, the lower weighing 4 lbs. and the upper 5 lbs., find what power will sustain a weight of 71 lbs. Also find through what height W will rise if the point of application of P be pulled vertically downwards through a distance of 4 feet.

3. Draw a diagram of a system of weightless pulleys where the power is to the weight as 1 : 12.

Note.—The prime factors of 12 being 2, 2, and 3, it is evident that the desired result will be obtained by combining two movable blocks of the first system with one of the fourth.

4. Draw a diagram of a system of weightless pulleys where the power is to the weight as 1 : 18.

5. You are given 3 pulleys and a supply of rope: arrange them so as to get the greatest possible mechanical advantage.

Note.—If the power be applied upwards, it is evident that they should be arranged according to the first system when a mechanical advantage of 8 is obtained; but if the power has to be applied downwards, the third system will be the best, the mechanical advantage under these circumstances being 7.

THE INCLINED PLANE.

137. The **Inclined Plane** is a rigid plane making an angle with the horizon called the *inclination of the plane.* Considered as a mechanical power the Inclined Plane is an example of a machine having a fixed surface capable of equilibrating any force perpendicular to itself. It is chiefly used for raising heavy bodies by dragging them up the slope of the plane. We shall at present only consider the cases where all the forces acting on the body lie in a plane perpendicular to the intersection of the inclined plane with the horizon, and therefore lying in a vertical plane which cuts the inclined plane along the line of greatest slope.

138. In this chapter the inclined plane is supposed to be perfectly smooth; and therefore the only force with which it can react against a body in contact with it is one perpendicular or normal to its surface which is called the *normal reaction* of the plane.

139. **Case I.**—To find the Conditions of Equilibrium when *the Power acts along or parallel to the plane.*

Let AB be the line of greatest slope of the plane through the point A (fig. 74); AC the horizontal line through A.

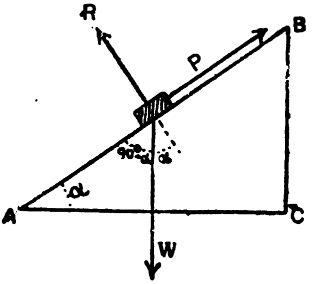

FIG. 74.

Now, if from a point B a perpendicular BC be dropped on AC, then BC is called the *height*, AC the *base*, and AB the *length* of the plane.

Let the inclination of the plane $BAC = \alpha$.

Let a body of weight W be supported on the plane by a force P acting along the plane: then the body is in equilibrium under the action of P, W, and the reaction R of the plane acting perpendicularly to the plane. Now *resolve these forces along the plane,* i.e. along the direction AB, and we have—

The *resolute* of P along $AB = P$ (upwards).

The *resolute* of R along $AB = 0$.

The *resolute* of W along $AB = W\cos(90-\alpha)$(downwards);

$$\therefore \quad P - W\cos(90-\alpha) = 0, \quad \text{or} \quad P = W\sin\alpha.$$

Again, *resolve the forces in a direction perpendicular to the plane*; now the resolute of $P = 0$, and we have—

$$R - W\cos\alpha = 0, \quad \text{or} \quad R = W\cos\alpha.$$

Ex. 1. The height of a smooth plane is to its base as 3 : 4; find what power acting parallel to the plane will support a weight of 15 lbs. Also find the pressure on the plane.

Here, resolving along the plane, we have—

$$P - 15 \cos (90 - \alpha) = 0,$$

or

$$P = 15 \sin \alpha = 15 \times \tfrac{3}{5} = 9 \text{ lbs. wt.}$$

Again, resolving perpendicular to the plane—

$$R - 15 \cos \alpha = 0,$$

or

$$R = 15 \cos \alpha = 15 \times \tfrac{4}{5} = 12 \text{ lbs. wt.}$$

Ex. 2. If the power be to the weight as $\sqrt{3} : 2$, find the inclination of the plane, the line of action of the power being parallel to the plane.

Let $P = \sqrt{3}$ and $W = 2$;

therefore, as before, $\sqrt{3} = 2 \sin \alpha$;

$$\therefore \sin \alpha = \frac{\sqrt{3}}{2} \qquad \therefore \alpha = 60°.$$

EXAMPLES XIX.

1. Verify the formulæ $P = W \sin \alpha$ and $R = W \cos \alpha$, where P acts parallel to the plane (*a*) by means of *Lami's Theorem*, (*b*) by means of the *triangle of forces*.

2. The height of a smooth plane is to its base as 5 : 12: find what power acting parallel to the plane will support a weight of 52 lbs. Also find the pressure on the plane.

3. The base of a smooth plane is to its length as 40 : 41: find what power acting parallel to the plane will sustain a weight of 82 lbs.

4. A weight W placed on a smooth inclined plane is sustained by a rope whose direction is parallel to the plane, and which, after passing over a smooth pulley at the top of the plane, hangs vertically downwards with a weight equal to $\frac{W}{2}$ suspended at its free end: find the inclination of the plane and the normal pressure of W on the plane.

5. If, when P is parallel to the plane, P is to W in the ratio 11 : 61, find the ratio of R to W.

140. **Case II.**—To find the Conditions of Equilibrium when *the Power acts horizontally*, i.e., *parallel to the base of the Plane.*

In this case the power applied to the body must be a *push* acting horizontally from the left-hand side (fig. 75), and by producing out the line of action of this force we may regard it as a *pull* acting towards the right-hand side.

Resolve the forces along the plane AB, and we have :—

The *resolute* of P along $AB = P \cos \alpha$ (upwards).

The *resolute* of R along $AB = 0$.

The *resolute* of W along $AB = W \cos (90 - \alpha)$ (downwards);

$$\therefore \quad P \cos \alpha - W \cos (90 - \alpha) = 0,$$

or $\quad P \cos \alpha = W \sin \alpha; \quad \therefore \quad P = W \tan \alpha.$

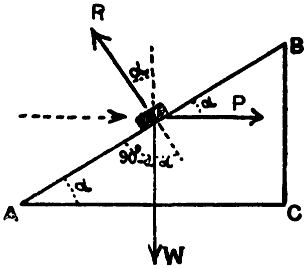

FIG. 75.

Again, *resolve the forces vertically*; now the resolute of $P = 0$, and we have

$$R \cos \alpha - W = 0, \text{ or } R = \frac{W}{\cos \alpha} = W \sec \alpha.$$

Note.—We could also have obtained the same value of R if we resolved in a direction perpendicular to the plane, as in the last case; but the method would be somewhat longer. The student is, however, advised, for the sake of the practice it gives him, to verify this himself.

Ex 1. The angle of inclination of a smooth plane is 60°: find what force acting horizontally will sustain a weight of 20 lbs. on the plane. Also find the normal reaction of the plane.

Here, resolving along the plane (fig. 75), we have :—

$$P \cos 60 - 20 \cos 30 = 0,$$

or

$$P \times \tfrac{1}{2} - 20 \times \frac{\sqrt{3}}{2} = 0, \qquad \therefore \quad P = 20\sqrt{3} \text{ lbs. wt.}$$

Again, resolving vertically :—

$$R \cos 60 - 20 = 0, \text{ or } \frac{R}{2} = 20, \quad \therefore \ R = 40 \text{ lbs. wt.}$$

Ex. 2. The height of a plane is to its length as 5 : 13 : find what horizontal force will support a weight of 48 lbs. on the plane. Also find the pressure on the plane.

Here, resolving along the plane :—

$$P \cos \alpha - 48 \cos (90 - \alpha) = 0, \quad \text{or} \quad P \cos \alpha = 48 \sin \alpha,$$

or
$$P = 48 \tan \alpha = 48 \times \frac{5}{12} = 20 \text{ lbs. wt.}$$

And resolving vertically

$$R \cos \alpha - 48 = 0; \quad \therefore \frac{R\,12}{13} = 48, \quad \therefore R = 52 \text{ lbs. wt.}$$

EXAMPLES XX.

1. Verify the formulae $P = W \tan \alpha$ and $R = W \sec \alpha$ where P is horizontal (*a*) by means of *Lami's Theorem,* (*b*) by means of the *triangle of forces.*

2. Find the inclination of a smooth plane when a body on the plane is sustained in equilibrium by a horizontal force equal to its own weight. Also find the normal pressure on the plane.

3. If, when the power is horizontal, W is to R as 1 : 2, find the inclination of the plane, and also find the relation of W to P.

4. In a smooth, inclined plane, if a weight of 120 lbs. is supported by a horizontal force of 22 lbs. wt., find the ratio of the height to the length: also find the normal pressure.

141. **Case III.**—To find the Conditions of Equilibrium when *the Power makes an angle* θ *with the Plane.*

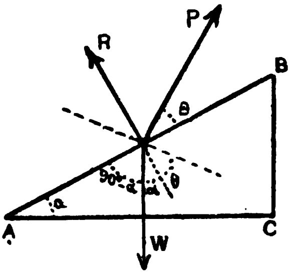

FIG. 76.

Resolve the forces along the plane AB.

The *resolute* of P along $AB = P \cos \theta$ (upwards).

The *resolute* of R along $AB = 0$.

The *resolute* of W along $AB = W \cos (90 - \alpha)$(downwards);

$\therefore \ P \cos \theta - W \cos (90 - \alpha) = 0$, or $P \cos \theta = W \sin \alpha$, or

$$P = \frac{W \sin \alpha}{\cos \theta}.$$

Again, *resolve the forces in a direction perpendicular to that of* P: now the resolute of $P = 0$.

The student will easily see that the direction of R makes an angle θ with the perpendicular to P, and that of W an angle $\alpha + \theta$ with the same perpendicular:

$$\therefore\ R\cos\theta - W\cos(\alpha+\theta)\,;\quad \therefore\ R = \frac{W\cos(\alpha+\theta)}{\cos\theta}.$$

It is evident that Case II can be deduced from this case by making $\theta = -\alpha$.

Note.—As before, this value of R can also be obtained by a somewhat longer method by either resolving vertically or perpendicularly to the plane.

Ex. 1. The height of a smooth plane is to its base as 11 : 60 : find what force, making an angle of 60° with the plane, will be sufficient to sustain a weight of 122 lbs.

Here $$P\cos 60 - 122\cos(90-\alpha) = 0,$$
or
$$P\,.\,\tfrac{1}{2} = 122\sin\alpha = 122 \times \frac{11}{61} = 22,\quad \therefore\ P = 44 \text{ lbs. wt.}$$

Ex. 2. A smooth inclined plane has a rise or gradient of 1 in 20 (this means that the height is to the length as 1 : 20). If a body be supported on the plane by three ropes which are pulled with forces of 6, 8, and 15 lbs. wt. respectively, and whose respective inclinations to the plane are θ_1, θ_2, and θ_3, find the weight of the body given $\cos\theta_1 = \frac{1}{3}$, $\cos\theta_2 = \frac{1}{4}$, and $\cos\theta_3 = \frac{1}{5}$.

Here, resolving along the plane,

$$6 \times \tfrac{1}{3} + 8 \times \tfrac{1}{4} + 15 \times \tfrac{1}{5} = \frac{W}{20};\quad \therefore\ W = 140 \text{ lbs.}$$

EXAMPLES XXI.

1. A weight of 43 lbs. on a smooth plane 13 feet long and 5 feet high is in equilibrium under the action of a force of 7 lbs. wt. up the plane, together with a force of $2P$ up the plane, and a force of $3P$ parallel to the base: find the magnitude of P and the normal reaction.

2. Two bodies whose weights are W and W' rest on two smooth planes whose angles of inclination are α and β respectively, being connected by a string passing over a smooth pulley placed at their common vertex. Prove that

$$W : W' :: \sin\beta : \sin\alpha,$$

i.e. that the weights are inversely as the sines of the angles of inclination of the respective planes on which they rest.

3. A heavy rope passes over a pulley placed at the common vertex of a double inclined plane, placed back to back, a portion of the rope resting on each plane. Prove that, if the rope is in equilibrium, the line joining its extremities is horizontal.

4. A bead moves on a smooth vertical circle made of wire, being tied to the highest point of the circle by a string equal in length to the radius of the circle: find the tension of the string, and the normal reaction of the wire when the bead is in equilibrium.

5. If R be the pressure on the plane when a body of weight W is supported by a force acting parallel to the plane, and R' the pressure when the force acts horizontally, show that $W = \sqrt{RR'}$.

6. The angle of inclination of a smooth plane is 45°. Find what force, making an angle of 30° with the plane, will support a body whose weight is 100 lbs. on it. Find also the normal reaction of the plane.

7. A weight of 30 lbs. is supported on a smooth plane whose inclination is $\sin^{-1}\frac{3}{5}$ by the conjoint action of a force of 12 lbs. wt. acting up the plane, and a force P acting horizontally: find the value of P, and the reaction of the plane.

The Wedge or Movable Inclined Plane.

142. A **Wedge** is a solid triangular prism used as a movable inclined plane, and is employed to separate bodies which are pressed together with very great forces, or to split a body, such as a block of stone, or a log of wood, by driving the thin end of the wedge into a cleft or hollow in the body.

143. The wedge generally may be regarded as a double inclined plane placed back to back.

Thus ABC represents a section of a wedge being driven into the cleft DEF by a force P applied perpendicularly to the plane face BC.

Let Q and R be the reactions at the points of contact D and F respectively, and, supposing the surfaces are smooth, these forces act at right angles to the sides of the wedge.

Then, for equilibrium, the direction of P must pass through O, the intersection of Q and R.

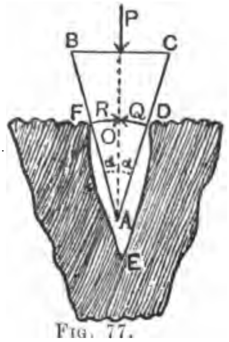

FIG. 77.

Now, since P, Q, and R, acting perpendicularly to the sides of the triangle ABC, are in equilibrium, they must each be proportional to the side to which it is perpendicular (Cor. 2, Art. 25);

$$\therefore \frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB},$$

or if the lengths of these sides be denoted by a, b, and c respectively,

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c},$$

or, if the wedge be isosceles, as is generally the case, then

$$b = c, \text{ and } Q = R;$$

$$\therefore \quad \frac{P}{a} = \frac{R}{b}; \quad \therefore \quad P = \frac{Ra}{b} = 2R \sin \alpha,$$

where 2α denotes the vertical angle of the wedge BAC.

Note.—In the case of the isosceles wedge the same result can be obtained by resolving along the direction of P, when we obtain

$$P = R \cos (90 - \alpha) + Q \cos (90 - \alpha) = 2R \cos (90 - \alpha) = 2R \sin \alpha.$$

144. It is evident since $R = \frac{Pb}{a}$ that, by increasing the length b, and by diminishing the breadth a, the value of R may be made as great as we please; and hence the longer and thinner the wedge is the greater will be the resistance which can be overcome by the power. The knife, the axe, and the chisel are examples of the principle of the wedge applied to practical purposes. In most cases, however, the power is applied, not by a steady pressure, but by one or more blows delivered by a hammer, or otherwise, against the face of the wedge.

The Screw.

145. The Screw may be regarded as a movable inclined plane wrapped round a cylinder.

Thus, if ABC be a triangular piece of paper right-angled at C, AC being horizontal, and $adef$ a circular cylinder such as a lead pencil, then, if, the corner A of the paper coinciding with a, and the direction AC with ad, we wrap the paper round the cylinder, the slanting side AB will trace out the spiral represented in the diagram (fig. 78), which represents the curve along which the thread of the screw runs.

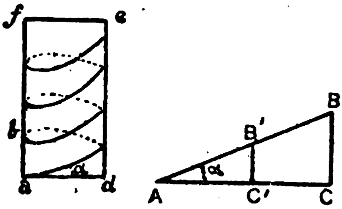

FIG. 78.

Again, if we cut off AC' equal to the circumference of the cylinder, and draw $C'B'$ parallel to CB, then the slip of paper $AC'B'$ will go exactly once round the cylinder, B' coinciding with b, and C' with a, while $B'C'$ will be equal to ba, the interval between the threads of the screw. This interval between two consecutive threads measured parallel to the axis is called the **pitch** of the screw.

The **angle** of the screw is the angle between the direction of the thread and the horizon, and is evidently the same at all parts of the thread, being equal to the angle of inclination α of the plane ABC.

Also $$\tan \alpha = \frac{B'C'}{AC'} = \frac{\text{interval between the threads}}{\text{circumference of the screw}},$$

or $$\tan \alpha = \frac{i}{2\pi a},$$

where i denotes the interval between the threads, and a the radius of the screw.

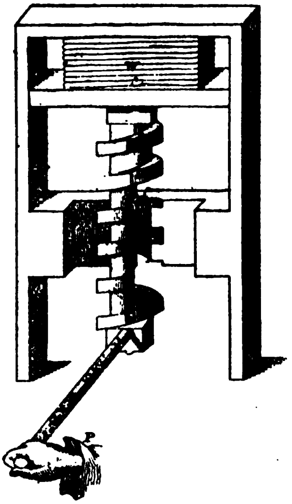

Fig. 79.

The raised thread of the screw is usually fitted into a corresponding grooved spiral cavity cut in the interior surface of a hollow cylinder called the companion screw (fig. 79).

We shall assume in our calculations that the thread of the screw is rectangular in shape, and that it works without friction in the groove of the companion screw into which it is fitted.

To find the Condition of Equilibrium when a Weight W is sustained by a Power P by means of a Screw.

146. Let the power P be applied horizontally at the extremity of a lever of length r, and let Q be the horizontal force acting at the extremity of the radius of the cylinder of the screw which would balance P; then by the principle of the wheel and axle we have, the radius of the cylinder of the screw being represented by a,

$$P \times r = Q \times a, \quad \text{or} \quad P \times 2\pi r = Q \times 2\pi a,$$

or $P \times$ circumference of circle described by P
$= Q \times$ circumference of the screw.

Again, the vertical weight W may be regarded as divided up into W_1, W_2, etc., applied to the separate parts of the thread. Similarly, Q may be divided up into corresponding portions Q_1, Q_2, etc.; therefore, a small portion of the thread at O (fig. 80) is in equilibrium under the forces Q_1, W_1, and R_1 (the normal reaction). Hence (see Case II of Inclined Plane) we have

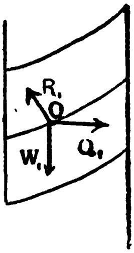

FIG. 80.

$$Q_1 = W_1 \tan a.$$

Similarly, $Q_2 = W_2 \tan a$, etc.

Hence, adding we obtain

$$Q_1 + Q_2 + \ldots = (W_1 + W_2 + \ldots) \tan a,$$

or
$$Q = W \tan a;$$

but
$$\begin{aligned} P \times 2\pi r &= Q \times 2\pi a \\ &= W \tan a \times 2\pi a \\ &= W \frac{i}{2\pi a} \times 2\pi a; \end{aligned}$$

$$\therefore \quad P \times 2\pi r = W \times i,$$

or $P \times$ circumference of its circle = $W \times$ interval between the threads, or *the power is to the weight as the interval between the threads is to the circumference of the circle described by the power.*

NOTE.—Another proof of this formula more easily grasped by the student is given in the chapter on Work.

Ex. If the length of the power-arm of a screw be 1¾ feet (this means the length of the radius of the circle described by the point of application of the power), find what power would support a weight of 55 lbs., the screw having 4 threads to the inch. The value of π may be taken as $\frac{22}{7}$.

Here the interval between two consecutive threads = ¼ inch.

Then, regarding the screw as a combined wheel and axle and inclined plane, we eventually obtain

$$P \times 2 \times \frac{22}{7} \times 21'' = 55 \times \frac{1}{4}, \ 132\,P = \frac{55}{4}; \quad \therefore \quad P = \frac{55}{4 \times 132} = \frac{5}{48} \text{ lb. wt.}$$

EXAMPLES XXII.

1. The length of the power-arm of a screw is 10 inches: find the weight raised by a power of 12 lbs. wt. if the distance between the threads is $\frac{1}{12}$ of an inch.

2. The circumference of the circle described by the power in the screw is 6 feet: find the weight sustained by a power of 10 lbs. wt., the screw having 10 threads to the inch.

3. If the interval between the threads of a screw be ⅛ of an inch, and the length of the power-arm be 3½ feet, find the mechanical advantage.

4. In a screw the length of the power-arm is 2 feet 4 inches, and the interval between the threads ¼ of an inch: find what power will support a weight of 1 ton.

5. If the interval between the threads be ¼ of an inch, and a power of $\frac{5}{24}$ lb. wt. can just sustain a weight of 110 lbs., find the length of the power-arm.

6. If P = ¼ lb. wt., find W when P acts 14 inches from the axis, and the screw advances 10 inches in 120 turns.

7. The mechanical advantage of a screw is 66; the power is applied at the circumference of the cylinder perpendicular to the radius of the cylinder; the diameter of the cylinder is 7 inches: find the interval between the threads, and also how many threads there are in an inch.

8. Show that the mechanical advantage of a screw is inversely proportional to the pitch, provided the length of the power-arm remains the same, i.e., the smaller the pitch the greater the mechanical advantage.

CHAPTER X.

FRICTION.

147. If all bodies were perfectly smooth, there would be no resistance experienced in moving the surface of one body over that of another in contact with it; or, in other words, the only reaction which could exist between two such bodies would be perpendicular or normal to their common surface.

However, there are no bodies in existence which are perfectly smooth, and hence there is always some resistance experienced in sliding the surface of one body over that of another. This resistance is called **Friction.**

148. **Limiting Friction.**—When two rough surfaces are in contact with one another, there is no friction between the surfaces, unless there is a tendency for one body to slide over the surface of the other. In all cases the force of friction is equal and opposite to the force which tends to produce the sliding motion; and hence, as the latter force is gradually increased, the force of friction will also increase until, when motion is just about to take place, it reaches its maximum amount; in this case the friction is called *limiting or statical friction,* which may be defined as *the friction between two bodies when one is on the point of sliding over the other.* Also the bodies are then said to be in *limiting equilibrium.*

149. The Laws of Friction were first investigated by a series of experiments by Coulomb in 1781. He showed that the resistance to be overcome in causing a body to

commence to move is greater than that experienced in keeping the body moving continuously, the former being called, as we have already seen, *statical friction*, and the latter *kinetic friction*, or friction of motion.

Laws of Limiting Friction.

1. *The amount of friction depends on the quality of the surfaces in contact.*

2. *The direction of the force of friction on each particle of the body in contact with the other is opposite to the direction in which that particle is on the point of moving.*

3. *The friction is directly proportional to the normal pressure.*

4. *Provided the normal pressure remains the same, the friction is independent of the extent of the surfaces in contact.*

5. *The friction depends on the duration of contact, and increases for a certain time until it reaches its greatest value, when it remains constant.*

6. *When motion takes place, the magnitude of the friction is independent of the velocity of motion.*

150. These statements are found by experiment to be sufficiently accurate for practical purposes, and may be regarded as true for those pressures and velocities which usually occur in practice; but beyond certain limits they are incorrect, and they may therefore be regarded rather as practical rules than "Laws" in the strict sense of the term.

151. **Coefficient of Friction.**—From the Third Law of Limiting Friction it follows that the limiting friction bears a constant ratio to the normal pressure, the value of this constant ratio depending on the nature of the surfaces in contact; it is generally denoted by μ and is called the *Coefficient of Friction*, which may therefore be defined as *the*

ratio of the friction to the normal pressure when motion is about to take place. Thus, if a body at O (fig. 81) be on the point of moving in the direction OA under the action of a force F acting in the direction OA, then the limiting friction, being equal and opposite to F, acts in the direction OB, the production of OA; and if R be the normal pressure between the two surfaces, we have

$$\frac{F}{R} = \mu, \quad \text{and} \quad \therefore \quad F = \mu R.$$

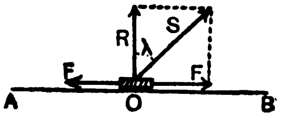

FIG. 81.

Ex. 1. If the normal pressure be 20 lbs. wt. and the coefficient of friction be $\frac{1}{4}$, find the limiting friction.

Here $F = \mu R$; $\therefore$ $F = \frac{1}{4} \cdot 20 = 5$ lbs. wt.

Ex. 2. If a mass of 50 lbs. wt. be placed on a horizontal table and a string attached to the body, and, passing over a smooth pulley at the edge of the table, carry a scale-pan at its extremity; then, if the scale-pan be gradually loaded with weights until motion is about to take place, find the coefficient of friction, given that the weights on the scale-pan together with the weight of the scale-pan itself amount to 20 lbs.

Here $F = \mu R$; $\therefore$ $20 = \mu \cdot 50$; $\therefore$ $\mu = \frac{20}{50} = \frac{2}{5}$.

152. **The Angle of Friction.**—If the normal reaction and the friction be compounded into a single force, this force is called the *total resistance*; and the angle which the direction of this force makes with the normal *when motion is about to take place* is called the *angle of friction.*

Thus the resultant force S (fig. 81) of the normal reaction R and the friction F is the total resistance, and the angle λ between the direction of S and the normal to the surface in contact is the angle of friction. Also

$$\tan \lambda = \frac{F}{R} = \mu.$$

Hence *the coefficient of friction is equal to the tangent of the angle of friction.* Thus if the coefficient of friction be $\frac{1}{\sqrt{3}}$, the angle of friction is 30°, for $\tan 30 = \frac{1}{\sqrt{3}}$.

153. **Cone of Friction.**—Let AO be a bar or rod, such as a walking-stick, pressed against a hard surface BC, as, for example, the surface of a frozen pool of water: then let the obliquity AON of the bar to the normal ON be gradually increased (N being any point on the normal) until motion is about to take place, then the maximum value of this angle is the angle of friction λ; then describing round ON a right cone OED, whose semi-vertical angle NOE is equal to λ, this cone is called the *cone of friction*, and, if the direction of the total resistance at O lie anywhere inside this cone, there will be equilibrium, and motion will not take place.

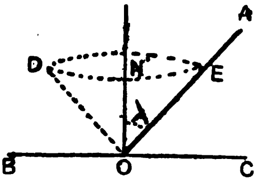

FIG. 82.

ROUGH INCLINED PLANE.

Where the Power acts parallel to the plane.

154. Let a weight W, on a rough inclined plane, be *on the point of moving up the plane,* under the action of a force P acting parallel to the plane. Then, since the body is on the point of moving up, the friction μR acts down the plane (R being the normal pressure). Now, *resolving the forces along the plane* (here R has no component), we have

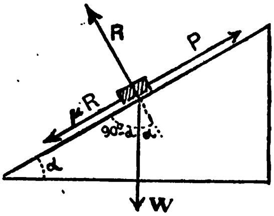

FIG 83.

$$P - W\cos(90 - a) - \mu R = 0,$$

or

$$P = W\sin a + \mu R.$$

Again, resolving the forces in a direction perpendicular to the plane (neither P nor μR having any component), we have

$$R - W\cos a = 0, \quad \text{or} \quad R = W\cos a;$$

$$\therefore \quad P = W\sin a + \mu W\cos a = W(\sin a + \mu\cos a).$$

Again, if the body be *on the point of moving down the plane*—that is, if the force P *just sustains* the body—then the friction μR acts up the plane; and we have

$$P - W \sin a + \mu R = 0,$$

and eventually $$P = W(\sin a - \mu \cos a).$$

Ex. 1. The angle of inclination of a rough plane is 30°; a body of mass 20 lbs. is in equilibrium on the plane under the action of a force acting parallel to the plane; if the coefficient of friction be

$$\frac{1}{2\sqrt{3}},$$

find the magnitude of the force (1) if the body be *on the point of moving* up the plane, (2) if the body be *just supported* on the plane.

Here, resolving along the plane (fig. 83), if μR acts down the plane,

$$P - 20 \cos 60 - \mu R = 0,$$

or $$P = 10 + \frac{1}{2\sqrt{3}} R.$$

Again, resolving perpendicularly to the plane,

$$R - 20 \cos 30 = 0, \quad \text{or} \quad R = 10\sqrt{3}.$$

$$\therefore \quad P = 10 + \frac{1}{2\sqrt{3}} 10\sqrt{3} = 15 \text{ lbs. wt.}$$

Also, if the body be *barely supported*, it is on the point of moving down; therefore μR acts up the plane,

$$\therefore \quad P - 20 \cos 60 + \mu R = 0,$$

and eventually

$$P = 10 - \frac{1}{2\sqrt{3}} 10\sqrt{3} = 5 \text{ lbs. wt.}$$

Ex. 2. The height of a plane is to its base as 9 : 40. A body of mass 82 lbs. rests on the plane, and is *just prevented from slipping down* by a force of 8 lbs. wt.: find the coefficient of friction.

Here, resolving along the plane,

$$8 - 82 \sin a + \mu R = 0,$$

or $$8 = 82 \times \frac{9}{41} - \mu R = 18 - \mu R.$$

Again, resolving perpendicularly to the plane,

$$R = 82 \cos a = 82 \times \frac{40}{41} = 80;$$

$$\therefore \quad 8 = 18 - \mu 80; \quad \therefore \quad 80\mu = 10; \quad \therefore \quad \mu = \frac{1}{8}.$$

Where P makes an Angle θ with the Plane.

155. Here, resolving along the plane (fig. 84), if the body is on the point of moving up the plane, we have

$$P\cos\theta - W\cos(90 - \alpha) - \mu R = 0,$$

or

$$P\cos\theta = W\sin\alpha + \mu R.$$

Again, resolving perpendicularly to the plane,

$$R + P\cos(90 - \theta) - W\cos\alpha = 0;$$

$$\therefore\quad R = W\cos\alpha - P\sin\theta.$$

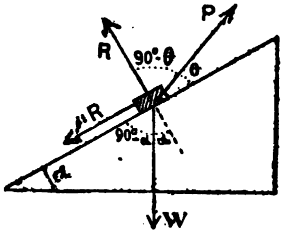

FIG. 84.

Substitute this value of R, and we have

$$P\cos\theta = W\sin\alpha + \mu(W\cos\alpha - P\sin\theta),$$

or

$$P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha);$$

$$\therefore\quad P = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\theta + \mu\sin\theta}.$$

Similarly, if P just supports the body, the friction acts up the plane; and we have

$$P = W\frac{\sin\alpha - \mu\cos\alpha}{\cos\theta - \mu\sin\theta}.$$

Note.—The former case where P acts parallel to the plane could have been deduced from this case by making $\theta = 0$.

156. *Where P acts horizontally.* By resolving along and perpendicularly to the plane as before, we eventually obtain

$$P = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\alpha - \mu\sin\alpha}\text{ just to drag up,}$$

or

$$P = W\frac{\sin\alpha - \mu\cos\alpha}{\cos\alpha + \mu\sin\alpha}\text{ just to sustain.}$$

These results can also be deduced from the previous case by making $\theta = -\alpha$.

Ex. A particle of mass 52 ozs. is on a rough inclined plane whose inclination is

$$\sin^{-1}\frac{5}{13}$$

$\left(\text{this means the angle whose sine is }\frac{5}{13}\right)$: what force making the angle $\sin^{-1}\frac{4}{5}$ with the plane would be just sufficient to cause the body to move up the plane, the coefficient of friction being $\frac{1}{4}$? Also find R.

Here, resolving along the plane (fig. 84),

$$P\cos\theta = 52\sin\alpha + \frac{1}{4}R, \quad\text{or}\quad \frac{P3}{5} = 52\times\frac{5}{13} + \frac{1}{4}R;$$

$$(1)\quad\text{or}\quad \frac{P3}{5} - \frac{1}{4}R = 20.$$

Again, resolving perpendicularly to the plane,

$$P\sin\theta + R = 52\cos\alpha;$$

$$(2)\quad\text{or}\quad \frac{P4}{5} + R = 48;$$

$$(1)\times 4\ldots\frac{P12}{5} - R = 80.$$

$$\therefore\ \text{Adding,}\ \frac{16P}{5} = 128;\quad \therefore\ P = 40\ \text{ozs. wt., and}\ R = 16\ \text{ozs. wt.}$$

If a body be in equilibrium on a rough inclined plane under the action solely of its own weight and the reaction of the plane, and if the inclination of the plane be gradually increased, the body will be on the point of sliding down when the inclination of the plane is equal to the angle of friction.

157. For, when on the point of slipping down, the body is in limiting equilibrium under the action of W, R, and μR (fig. 85). But R and μR are equivalent to the total resistance S whose direction makes the angle of friction λ with R.

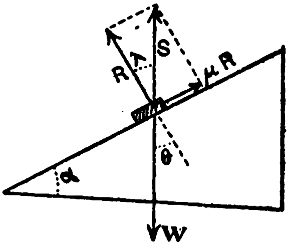

FIG. 85.

Therefore S and W, being in equilibrium, are opposite in direction; hence the angles λ and θ are equal (I. 15); but $\theta = \alpha$; $\therefore\ \alpha = \lambda$, or the inclination of the plane is equal to the angle of friction.

Thus if, when the body is on the point of sliding down, the height of the plane is to its base as 5 : 12, we conclude that the angle of friction

$$\lambda = \tan^{-1} \frac{5}{12},$$

and the coefficient of friction

$$\mu = \tan \lambda = \frac{5}{12}.$$

Ex. If a body of mass 40 lbs. is in limiting equilibrium on an inclined plane whose height is to its base as 3 : 4, find what force acting parallel to the plane will just sustain it if the angle of inclination be increased until the height is to its base as 4 : 3.

Here $\mu = \tan \lambda = \frac{3}{4}$.

Resolving along and perpendicularly to the plane we obtain the two equations :

$$P = 32 - \tfrac{3}{4} R \quad \text{and} \quad R = 24\,; \quad \therefore \quad P = 14 \text{ lbs. wt.}$$

Note.—The coefficient of friction between any two substances can be found by means of the theorem in the last article ; a flat slab of one substance is placed on a plane made of the other substance ; and, on the inclination of the plane being increased until motion just takes place, the ratio of the height of the plane to its base will give $\tan \lambda$ or μ, the coefficient of friction.

EXAMPLES XXIII.

1. The ink-bottle on the table at which I write weighs 10 ozs., and the coefficient of friction between the glass of the bottle and the polished mahogany of the table is ·3 : find what horizontal force will be sufficient to just move the bottle along the table.

2. A sledge whose weight, including its occupants, is 60 stone, is at rest on a frozen lake. The coefficient of friction between the polished steel of the runners and the ice is ·12 : find what horizontal force is necessary just to cause it to move, and find the magnitude of the resultant reaction of the ice.

3. Why is it that in walking along ice or any very slippery plane surface we take very short steps, and keep the direction of our legs almost perpendicular or normal to the surface of the ice ?

4. A body whose weight is 20 lbs., placed on a rough horizontal plane, is in limiting equilibrium under the action of a force of 10 lbs. wt. inclined to the horizon at an angle of 30° : find the coefficient of friction.

5. A body of weight 20 lbs. on a rough horizontal plane is about to slide along the plane under the action of a force whose inclination to the horizon is 45° : find the magnitude of the force, the value of μ being ·3.

6. A plane rises 5 in 13 (i.e. the inclination is $\sin^{-1} \frac{5}{13}$). Find what force, acting parallel to the plane, will be sufficient (1) just to drag up, (2) just to support, a body whose weight is 52 lbs., the coefficient of friction being $\frac{1}{4}$.

7. The angle of inclination of a rough plane is 60°: a body of mass 40 lbs. is in equilibrium on the plane under the action of a force acting parallel to the plane. If the coefficient of friction be ·5, find the magnitude of the force (1) if the body be on the point of *moving up* the plane, (2) if it be *just supported*.

8. A body of weight W is in equilibrium on a rough inclined plane of inclination α under the action of a force P whose direction makes an angle θ with the plane: show that

$$P = W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$$

when the body is on the point of moving up the plane, and that

$$P = W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

when the body is just supported, λ being the angle of friction.

9. A body of mass 156 lbs. just rests on an inclined plane where the height is to the base as 5 : 12 : find what force acting parallel to the plane will (1) just sustain, (2) just drag up, the body, supposing in each case the angle of inclination to be increased until the height is to the base as 12 : 5.

10. A particle of mass 45 ozs. is on a rough inclined plane whose inclination is $\sin^{-1}\frac{3}{5}$, and is acted on by a force of 10 ozs. wt. parallel to the plane: what additional force acting parallel to the base of the plane is required just to prevent the particle from slipping down, the coefficient of friction being $\frac{1}{3}$?

11. A uniform beam AB, 26 feet long, weighing 34 lbs., rests with one end A on a rough horizontal plane AC, and the other end B against a rough vertical wall CB. If AC is 10 feet, and if the coefficient of friction for both the plane and wall is $\frac{1}{3}$, find where a weight of 416 lbs. must be attached to the beam in order that the beam may be on the point of slipping down.

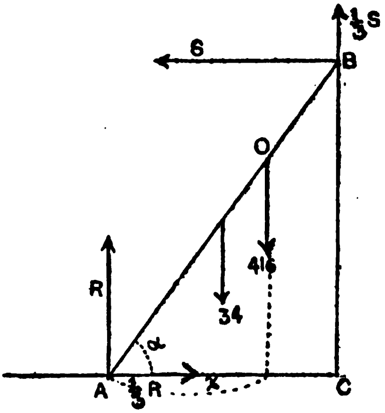

FIG. 86.

Here, if R and S be the normal reactions of the plane and wall respectively, the forces of friction are $\frac{1}{3}R$ and $\frac{1}{3}S$.

Let the 416 lbs. be placed at O, and let x be the length of the perpendicular from A on the vertical through O.

Now, resolving the forces vertically and horizontally, we obtain the two equations

$$R + \tfrac{1}{3}S - 34 - 416 = 0, \quad \text{and} \quad \tfrac{1}{3}R - S = 0,$$

Solving these equations, we obtain $R = 405$ and $S = 135$.

Again, taking moments round A, we have

$$24S + \frac{10S}{3} - 416x - 5 \times 34 = 0,$$

and, substituting the value of S, we get $x = \dfrac{110}{13}$; but

$$\frac{x}{AO} = \cos\alpha = \frac{10}{26}; \quad \therefore \; AO = \frac{110}{13} \times \frac{26}{10} = 22 \text{ feet}:$$

hence the weight is hung at a point of the ladder 22 feet from A.

12. A uniform rectangular block, 12 feet long, by 6 feet broad, by 6 feet high, weighing 520 lbs., on a rough plane inclined at $\sin^{-1}\frac{5}{13}$ to the horizon, with its longest side parallel to the line of greatest slope, is about to be pulled up the plane by a force acting parallel to the line of greatest slope at the centre of the upper side of its upper face: find the magnitude of this force, and also the point at which the resultant reaction of the plane is exerted, taking the coefficient of friction to be $\frac{1}{2}$.

13. In Example 12, find the least length of the rectangular block of this cross-section which can be pulled up the plane in such a way that one face continues to touch the plane.

14. AB, BC are two perpendicular rough planes, and AB is inclined at $\sin^{-1}\frac{3}{5}$ to the horizon. A uniform beam AC, 13 feet long, weighing 58 lbs., rests on the plane at A and C so that AB is 5 feet. Taking the coefficient of friction for each contact to be $\frac{1}{2}$, find the force which must be applied at A, in the direction BA, in order to move the beam.

15. Prove that when dragging a body either along a rough horizontal plane or up a rough inclined plane the body can be moved with the least expenditure of force when the direction of the force makes an angle with the plane equal to the angle of friction—i.e. the best angle of traction is the angle of friction.

16. Why can the gradients for railways not be made as steep as those for ordinary roads?

17. A uniform ladder, 13 feet long and weighing 50 lbs., is placed with one end against a rough vertical wall, and with the other end on the ground, its inclination to the vertical being $\sin^{-1}\frac{5}{13}$. If the ladder is just about to slip down, and the coefficient of friction between the ladder and the wall is the same as that between the ladder and the ground, find this coefficient, and also the normal reaction at the wall and ground.

18. A ladder AB, 15 feet long, whose centre of gravity divides it into two segments, of which the lower AG is 6 feet long, rests with one end A on the ground and the other end B against a vertical wall. If the coefficients of friction for the ground and wall are $\frac{1}{2}$ and $\frac{1}{3}$ respectively, find the inclination of the ladder to the horizon when the ladder is just about to slip.

PART II.

DYNAMICS.

CHAPTER XI.

VELOCITY AND ACCELERATION.

158. When a particle is in motion, the line traced out by its successive positions at each instant is called the *path* of the particle. This path may be a straight line, as is the case if the body is falling freely under the action of gravity, or it may be a curve such as is described by a body in the air when projected up in any direction not vertical. At present we shall only consider Rectilinear Motion—that is, motion in a straight line.

159. **Definition.**—The *Velocity* of a point is its rate of change of position.

The velocity of a point is said to be *uniform* when it passes over equal spaces in equal intervals of time—however small those intervals of time may be—and *variable* when this is not the case.

The velocity of a point is completely determined when we know (1) its magnitude, and (2) its direction. In order that the magnitude may be measured, some unit of velocity must be agreed upon. In the British Isles the unit of velocity usually adopted is a velocity of *one foot per second* denoted by 1 *f/s.* This unit is sometimes called a *velo.* In scientific work, however, the unit invariably used is a velocity of *one centimetre per second,* to which the name one *kine* has been given by a resolution of a committee of the British Association.

160. **Uniform Motion in a Straight Line.**—If a point move with a uniform velocity u for t units of time, it is evident, as it moves over u units of space in each unit of time, that the total distance traversed is ut. Denoting this distance by s, we have therefore in all cases of uniform rectilinear motion $s = ut$.

161. Graphically, this can be represented thus:—Lay down *AB* containing as many units of length as *t* represents units of time. Erect *AC* perpendicular to *AB* and containing *u* units of length; then completing the rectangle *AD*, the area of this rectangle (being = *ut*) contains the same number of units of area as the number of units of space traversed by the point.

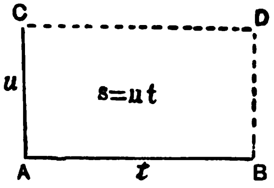

FIG. 87.

Ex. 1. A point moves with a uniform velocity of 20 *f/s*: find how far it will move in 5 seconds.

Here as it moves through a distance of 20 feet in each second, it will move through 100 feet in 5 seconds.

Ex. 2. A train moving with uniform velocity travels a distance of 60 miles in one hour: find its velocity in feet per second.

As it moves uniformly through a distance of 60 × 1760 × 3 feet in 60 × 60 seconds, it must therefore move through a distance of 88 feet in each second; hence its velocity is 88 *f/s*.

Ex. 3. Given the velocity of sound to be 1120 feet per second, and that the time which elapses between seeing a flash of lightning and hearing the thunder is 5 seconds, find the distance of the thunder-cloud from which the discharge took place, assuming that the propagation of light is instantaneous. *Ans.* 5600 feet.

162. **Mean or Average Velocity.**—When a point is moving with variable velocity for a definite interval of time, the *mean* or *average velocity* is that uniform velocity with which the body would have to move in order to describe the same space in the same interval of time.

The mean velocity during any interval may therefore be found by dividing the total number of units of space described by the number of units of time taken to describe it. For example, if a train travels a distance of 60 miles in one hour (including stoppages) with variable velocity,

the mean velocity = 60 × 1760 × 3 ÷ 60 × 60 = 88 *f/s*.

163. **Velocity at any Instant.**—When a point is moving with variable velocity, the *velocity at any instant* is the mean velocity for an infinitely small interval of time commencing

at that instant. Thus, if t represent the small interval of time (during which we may regard the velocity as uniform), and s the small space described during that interval, then

$$\text{the velocity } v = \frac{\text{the small distance } s}{\text{the small interval } t}.$$

Uniformly Accelerated Motion.

164. **Acceleration.**—When the velocity of a body is changing, *the rate of change in the velocity is called the acceleration.*

When equal changes of velocity take place in equal intervals of time, however small these intervals may be, the acceleration is *uniform,* and *variable* when this is not he case.

Note.—It is to be borne in mind that by "*change in velocity*" may be meant an alteration either in its magnitude or direction, and that in either case the motion is said to be accelerated. Thus a stone falling vertically downwards is an example of accelerated motion in which the magnitude of the velocity changes, but not its direction. On the other hand, it is equally true that a point describing a circle with uniform velocity is moving with accelerated motion, for although the magnitude of the velocity remains constant, its direction is changing at each instant.

The change in the velocity of a point moving in a straight line may be either an increase or a decrease; in the former case the acceleration is *positive,* and in the latter *negative.* A negative acceleration is called a *retardation.*

165. **Definition.**—The Unit of Acceleration is the acceleration of a point whose velocity increases by the unit of velocity in the unit of time.

In the F.P.S. system the Unit of Acceleration is a velocity of one foot per second added each second, and is usually termed an acceleration of "one foot per second per second," or 1*f*/*s*/*s*. This unit is sometimes called a *celo.*

In the C.G.S. system the Unit of Acceleration is a velocity of one centimetre per second added each second, usually termed an acceleration of "one centimetre per second per second," or 1 *c/s/s*.

In the present work the acceleration is supposed to be uniform unless the contrary is stated.

Ex. 1. A particle moving in a straight line has velocities 10, 12, and 14 feet per second respectively at the commencements of three consecutive seconds: what is the acceleration? *Ans.*— + 2 feet per sec. per sec.—i.e. 2 *f/s/s*.

Ex. 2. A particle moving in a straight line has velocities 12, 9, and 6 feet per second respectively at the commencements of three consecutive seconds: find the acceleration. *Ans.*— – 3 feet per sec. per sec.—i.e. – 3 *f/s/s*.

Ex. 3. A body projected vertically upwards with a velocity of 80 feet per second has a velocity of 48 feet per second at the end of one second; find the acceleration to which the body is subject. *Ans.*— – 32 *f/s/s*.

Uniformly Accelerated Rectilinear Motion.

166. Let a point moving in a straight line have a velocity u at the beginning of a certain interval of t units of time; let it be subject during that interval to a certain uniform acceleration f in the direction of its motion; if v represent the velocity at the end of the time t, and s the space gone over during that time, then

$$\begin{aligned} &(1) \quad \mathbf{v = u + ft;} \\ &(2) \quad \mathbf{s = ut + \tfrac{1}{2} ft^2;} \\ &(3) \quad \mathbf{v^3 = u^2 + 2fs.} \end{aligned}$$

167. **Proof of (1).**—Since f represents the increase in the velocity during each unit of time, we have (taking the unit of time as 1 second)

The initial velocity $= u$.
∴ The velocity at the end of 1 second $= u + f$;
the velocity at the end of 2 seconds $= u + 2f$;
the velocity at the end of 3 seconds $= u + 3f$.
∴ The velocity at the end of t seconds $= u + tf$.

$$\therefore \quad v = u + ft.$$

168. **Proof of (2).**—Since the velocity increases *uniformly* throughout the given time t, it follows that the velocity with which the point is moving when half this time

$$\left(\text{viz. } \frac{t}{2}\right)$$

has elapsed is the mean of its velocity at *any number* of units of time *before* the mid-interval, and that which it has at the *same number* of units of time *after* the mid-interval.

Hence, the velocity at the end of the time

$$\frac{t}{2}$$

is half the sum of the initial and final velocities; and this is the mean velocity of the point during the whole interval.

$$\therefore \quad \text{Mean velocity} = \tfrac{1}{2}(u + v) = \tfrac{1}{2}(u + u + ft) = u + \tfrac{1}{2}ft.$$

The point may now be supposed to have moved *uniformly* with this velocity during the interval t; therefore (Art. 160)

$$s = (u + \tfrac{1}{2}ft)\,t, \text{ or } s = ut + \tfrac{1}{2}ft^2.$$

This equation might also be written

$$s = \tfrac{1}{2}(u + v)\,t.$$

169. **Proof of (3).**—We have from (1)

$$v = u + ft.$$

Square both sides. $\therefore \quad v^2 = u^2 + 2uft + f^2t^2$

$$= u^2 + 2f(ut + \tfrac{1}{2}ft^2);$$

but by (2) $\qquad ut + \tfrac{1}{2}ft^2 = s. \quad \therefore \quad v^2 = u^2 + 2fs.$

170. The equation $s = ut + \frac{1}{2}ft^2$ can also be proved *graphically* as follows:—Lay down the line AB to represent the time t as in Art. 161. Erect AC perpendicular to AB, and cut off AC to represent the initial velocity u. Complete the rectangle AD; then the area of this rectangle represents (Art. 161) ut, or the space through which the point would move if it maintained its initial velocity unchanged throughout the interval.

Produce BD to E so that BE represents on the same scale the final velocity $u + ft$, DE representing the increase ft in the velocity during the t units of time. Join CE, and divide AB into t equal parts, AF, FG, GH, etc., each of these representing a unit of time. Erect perpendiculars FK, GL, etc., to AB at the points of division.

Now, since CD is divided by parallels into t equal parts, and DE represents ft; therefore XK represents f, and FK the velocity $u + f$ at the end of one unit of time. Similarly GL represents $u + 2f$, the velocity at the end of 2 units of time, etc

Again, if the velocity AC continued uniform during the interval of time AF, the space described would be represented by the rectangle $ACXF$ (Art. 161); and if during the same interval AF the point had moved uniformly with the velocity FK, the space described would be represented by the rectangle $ARKF$; but the velocity between A and F is greater than AC and less than FK: hence the space actually described in that interval is greater than one rectangle and less than the other, and the same is true of each of the other intervals; therefore the entire space described in the time AB is greater than the sum of the rectangles AX, FS, GT, etc., which all lie within the area $ACEB$, and less than the sum of the rectangles AK, FL, GM, etc., which exceeds the same area; and this is true *however small the intervals AF, FG, etc., may be*; but it is evident that the only space which is *always* greater than the sum of the inner and less than the sum of the outer rectangles is the area $ACEB$ itself.

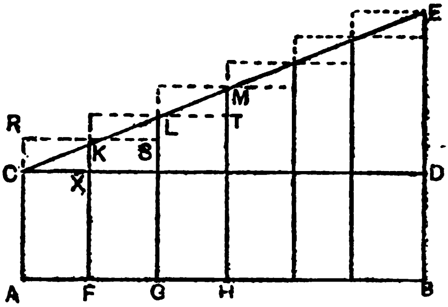

FIG. 88.

But $ACEB$ = rectangle AD + triangle CDE

$$= ut + \tfrac{1}{2} CD \, . \, DE$$

$$= ut + \tfrac{1}{2} t \, . \, ft,$$

or $$s = ut + \tfrac{1}{2} ft^2.$$

171. **Moving Point starting from Rest.**—If the moving point has no initial velocity—i.e. if at the commencement of the interval t it *starts from rest*—the formulæ in Art. 166 become, after making the substitution $u = 0$,

$$\mathbf{v} = \mathbf{ft},$$

$$\mathbf{s} = \tfrac{1}{2}\mathbf{ft}^2,$$

$$\mathbf{v}^2 = \mathbf{2fs}.$$

N.B.—In some of the following examples on uniformly accelerated motion, although the results can be arrived at more rapidly by the direct application of the formulæ in Art. 166, still the student will be greatly benefited if, in the case of the simpler examples, he works directly from first principles.

Ex. 1. A particle, starting with a velocity of 40 feet per second, is subject to a uniform retardation of 4 feet per second per second: find its velocity at the end of 6 seconds.

Here its velocity diminishes uniformly at the rate of 4 feet per second in each second; hence at the end of 6 seconds its velocity has diminished by $24 f/s$, and is therefore $40 - 24 = 16 f/s$.

Ex. 2. A train, whose speed is 30 miles per hour when steam is shut off, is brought to rest in 22 seconds: find the acceleration, which is supposed to be uniform.

$$30 \text{ miles per hour} = 30 \times 1760 \times 3 \div 60 \times 60 = 44 f/s,$$

and since it is brought to rest in 22 seconds;

$\therefore$ in 22 seconds its velocity has decreased by 44;

$\therefore$ in 1 second its velocity has decreased by 2;

hence the acceleration is $= -2$ feet per second per second.

The minus sign here means that it is a retardation.

Ex. 3. The speed of a train is reduced from 30 to 15 miles per hour while passing over a distance of half a mile: find the acceleration; also find how much further the train will travel before coming to rest.

Here $u = 44 f/s$, $v = 22 f/s$, and $s = 880$ yards $= 2640$ feet.

Now $$v^2 = u^2 + 2fs:$$

$$\therefore 22^2 = 44^2 + 2f \times 2640; \quad \therefore f = -\frac{11}{40} f/s/s \text{ (a retardation).}$$

Again, the initial velocity for the second period is 22, and the final velocity $= 0$;

$$\therefore \quad 0 = 22^2 + 2\left(-\frac{11}{40}\right)s,$$

or $$0 = 484 - \frac{11s}{20}; \quad \therefore s = 880 \text{ feet.}$$

Ex. 4. A body, starting with a velocity of 40 feet per second, is subject to a retardation of 4 feet per second per second: find the distance gone over in 6 seconds.

Here the velocity diminished by $4 f/s$ in each second;

$\therefore$ the velocity diminishes by $24 f/s$ in six seconds;

$\therefore$ the velocity at the end of 6 seconds $= 40 - 24 = 16$.

$\therefore$ the mean or average velocity $= \dfrac{40 + 16}{2} = 28$;

but space $=$ mean velocity $\times$ time $= 28 \times 6 = 168$ feet.

Otherwise, by the formula

$$u = 40, \quad f = -4, \quad t = 6,$$

$$s = ut + \tfrac{1}{2}ft^2 \quad \text{or} \quad s = 40 \times 6 + \tfrac{1}{2}(-4)6^2$$
$$= 240 - 72 = 168 \text{ feet.}$$

To find the Space described in a particular Second.

172. Let us suppose it is required to find the space described in the nth second by a body moving with uniform acceleration.

Now it is evident that the space gone over (say, in the 6th second after starting) = the space gone over in the first 6 seconds − that gone over in the first 5 seconds.

So similarly:—

The space in the nth second = space in n seconds − space in $(n-1)$ seconds: or by (2), Art. 166,

$$= \left[un + \tfrac{1}{2}fn^2\right] - \left[u(n-1) + \tfrac{1}{2}f(n-1)^2\right]$$
$$= un + \tfrac{1}{2}fn^2 - u(n-1) - \tfrac{1}{2}f(n^2 - 2n + 1),$$

or
$$s = u + \tfrac{1}{2}f(2n-1).$$

If the body starts from rest, then $u = 0$; and the formula then becomes $s = \frac{1}{2}f(2n-1)$.

173. **Corollary.**—From this it follows that, when *a body starts from rest, moving with a uniform acceleration,* the *spaces described in successive seconds are proportional to the odd numbers 1, 3, 5, etc.*

For, in the formula $s = \frac{1}{2}f(2n-1)$, substitute values 1, 2, 3, etc., for n; and we find that the spaces described in the 1st, 2nd, 3rd, etc., seconds are

$$\tfrac{1}{2}f,\ \tfrac{1}{2}f \times 3,\ \tfrac{1}{2}f \times 5, \text{etc.},$$

respectively, which are evidently proportional to 1, 3, 5, etc.

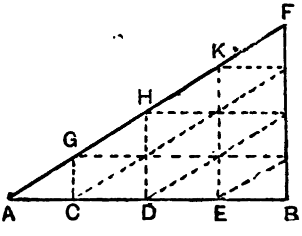

FIG. 89.

This result can also be shown beautifully by using the graphic method of Art. 170.

AB represents the time, as before; and *AC*, *CD*, etc., each represents one unit of time. The perpendicular *BF* represents the final velocity; then, as before,

the space in the 1st second = the triangle *ACG*,
" " 2nd second = the area *CGHD*;

but it is evident by drawing parallels that *CGHD* can be divided into three triangles each equal to *ACG*.

Similarly the area *DHKE* can be divided into five triangles each equal to *ACG*, and so on for the other spaces.

Ex. A body moving with uniform acceleration passes over 22 feet in the 4th second, and 6 feet in the 8th second, of its motion; find the initial velocity and the acceleration.

Here, distance in 4th second = distance in 4 seconds − distance in 3 seconds,

or $$22 = (u \,.\, 4 + \tfrac{1}{2}f \,.\, 4^2) - (u \,.\, 3 + \tfrac{1}{2}f \,.\, 3^2).$$

Similarly $$6 = (u \,.\, 8 + \tfrac{1}{2}f \,.\, 8^2) - (u \,.\, 7 + \tfrac{1}{2}f \,.\, 7^2).$$

These equations reduce to

$$22 = u + \tfrac{1}{2}f \,.\, 7,$$

and $$6 = u + \tfrac{1}{2}f \,.\, 15;$$

∴ subtracting $$16 = -\tfrac{1}{2}f8, \quad \text{or} \quad 16 = -4f;$$

∴ $f = -4$ *f/s/s* (a retardation). Also $u = 36$ *f/s*.

The value of f can be found otherwise, thus:—

The velocity in the middle of the 4th second, i.e. 3½ seconds after starting, being the average velocity during that second, must = 22 *f/s*.

Similarly the velocity 7½ seconds after starting = 6 *f/s*.

Hence, in 4 seconds its velocity has diminished by 16 feet per second, and therefore the acceleration is − 4 *f/s/s*.

EXAMPLES XXIV.

1. A particle starting with a velocity of 10 feet per second moves with an acceleration of 4 feet per second per second: find its velocity (*a*) at the end of 1 second, (*b*) at the end of 2 seconds, (*c*) at the end of 1 minute.

2. A particle starting with a velocity of 10 feet per second is subject to a retardation of 4 feet per second per second: find its velocity (*a*) at the end of 1 second, (*b*) at the end of 2½ seconds.

3. If in Ex. 1 the body started from rest, find what velocities it would have at the end of 1 second, 2 seconds, and 1 minute respectively.

4. A point has an initial velocity of 11 *f/s*, and moves with an acceleration 3 *f/s/s*: after what interval of time will its velocity be 30 miles per hour?

5. A train is moving with a velocity of 60 miles per hour when steam is shut off; if the resistance due to the action of the brakes, etc., produce a constant retardation of 4 *f/s/s*, how long will the train continue to move before coming to rest?

6. A body starting from rest moves with an acceleration of 4 *f/s/s*: find the distance traversed, and also its velocity at the end of 5 seconds.

7. A body having an initial velocity of 10 *f/s* moves with uniform acceleration and describes 100 feet in 5 seconds: find the acceleration.

8. What initial velocity must be given to a body in order that, moving with an acceleration of 32 *f/s/s*, it may describe 300 feet in 4 seconds?

9. A point starting from rest moves with an acceleration of 32 *f/s/s*: what distance will it move over in 1 second?

10. If a body starting with a velocity of 30 miles per hour, and subject to a uniform acceleration, describes 8 feet in 2 seconds, find the acceleration.

11. The velocity of a point is increased from 10 *f/s* to 15 *f/s* while it moves over 20 feet: find the rate at which its velocity is increased.

12. The velocity of a train is reduced from 60 miles per hour to 30 miles per hour while the train travels a distance of 121 yards: find (*a*) the acceleration; (*b*) how much further the train will travel before coming to rest.

13. Prove the formula $s = \frac{1}{2}(u + v)t$. A train is observed at two different instants to have velocities of 60 miles per hour and 40 miles per hour respectively; if the interval of time be 1 minute, find the distance traversed during the interval, supposing the rate of change in the velocity to be uniform.

14. A point starting with a velocity of 80 feet per second is brought to rest after it has moved over 100 feet: find the rate of change of velocity during the motion.

15. A body starting from rest moves over 96 metres in 8 seconds with uniform acceleration: how long will it take to acquire a velocity of 60 metres per second? and find the space described in that time.

16. A point starting from rest moves with an acceleration of 16 *f/s/s*: find the space traversed in the 9th second of its motion.

17. A point having an initial velocity of 20 feet per second is subject to a retardation of 2*f/s/s*: find how far it moves in the interval of time between the end of the third and the beginning of the fifth second.

18. A body moving with uniform acceleration passes over 25 and 49 feet in the 3rd and the 7th seconds of its motion respectively: find the initial velocity and the acceleration.

19. A body starting with a velocity of 30 *f/s*, and subject to a retardation of 4*f/s/s*, passes over 4 feet in a particular second: how long previous to the commencement of this second was the body in motion?

20. A body describes 20 metres in the 7th second of its motion, 26 metres in the 9th second, and 41 metres in the 14th second: is this inconsistent with the supposition that its motion is uniformly accelerated?

Falling Bodies.

174. A body falling under the action of gravity is an example of uniformly accelerated motion. When a body is let drop from a height above the surface of the Earth, it is evident to any observer that at first it moves slowly, but that, as it falls, there is a continuous increase in its velocity, or, in other words, it moves with an acceleration. That this acceleration is uniform will be shown in a subsequent chapter; at present we shall assume that such is the case. This acceleration is called the "*acceleration of gravity*," and *is the same for all bodies.*

175. To state that all falling bodies should have the same acceleration may at first appear ridiculous; for it may be said that, if a bullet and a feather be let fall through the same height, surely it is a matter of common experience that the bullet will reach the ground in a much shorter time, and must therefore fall with a much greater acceleration than the feather. This is indeed quite true; but we must remember that, as a body falls, the air displaced by it in its descent offers a resistance to its motion, and that this resistance is comparatively greater in the case of a light body such as a feather, which occupies a larger space and presents a larger surface to the air in proportion to its mass than a body of greater density such as a bullet. If, however, the bullet and feather be allowed to fall through the same height in a closed vessel from which the air has been exhausted by means of an air-pump, it is found that they reach the bottom of the vessel in exactly the same time, and that therefore they must have fallen with the same acceleration.*

176. *Value of Acceleration of Gravity.*—The acceleration of gravity is usually denoted by the letter "g." The value of g is always the same at the same place, but varies slightly at different parts of the Earth's surface. It is greatest at the Poles (Art 10), where its value is 32·25 *f/s/s*, and least at the Equator, where it amounts to 32·09 *f/s/s*.

At London at sea-level $g = 32{\cdot}19$ *f/s/s*, or in the C. G. S. system 981 *c/s/s*.

Note.—In all subsequent examples the resistance offered by the air to the motion of the body is neglected, and, unless the contrary is stated, the value of g is taken as 32 *f/s/s*.

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* The student can approximately arrive at the same result by the following simple experiment. Take a penny, and, holding it horizontally, place upon it a small piece of paper. Then let both drop, taking care that in its fall the penny remains horizontal. They will both be found to reach the ground in the same time. The penny, in its fall, clearing the air out of the way of the paper, the paper therefore falls as if it were in a vacuum. The student will see that this is only a rough experiment, as the conditions are complicated by the inrush of air behind the penny during its fall.

177. **Equations of Motion.**—If in Art. 166, g be substituted for f, the equations become—the body being supposed to be *projected vertically downwards* with an initial velocity u:—

$$\begin{aligned} v &= u + gt, \\ s &= ut + \tfrac{1}{2}gt^2, \\ v^2 &= u^2 + 2gs. \end{aligned}$$

178. On the other hand, when the body is *projected upwards* with an initial velocity u, the acceleration of gravity is a retardation, and $-g$ is therefore to be substituted for f; and the formulæ become

$$\begin{aligned} v &= u - gt, \\ s &= ut - \tfrac{1}{2}gt^2, \\ v^2 &= u^2 - 2gs. \end{aligned}$$

In this case we regard an upward direction as positive and a downward as negative. As the body ascends v decreases until, when the body reaches its highest point, it remains stationary for an instant when $v = 0$. The body then begins to move downwards; and during its fall v is negative.

179. It is of great importance that the student should bear in mind that when a body is projected upwards the letter s means *the height above the point of projection at any instant*, and not the actual distance moved through; for instance, if a body, on being thrown vertically upwards, ascends a distance of 100 feet, and then begins to fall, when it has fallen 20 feet the value of s at that instant is 80 feet, and not 120 feet; this follows from the fact that the 100 feet upwards is positive, and the 20 feet downwards is negative.

Body Projected vertically upwards.

180. **To find the time of ascent.**—When the body reaches its highest point, $v = 0$; and, substituting this value of v in the equation $v = u - gt$, we have

$$0 = u - gt; \quad \therefore t = \frac{u}{g},$$

which is the time of ascent. It can easily be shown that the time of ascent is equal to the time of descent. For, when the body reaches the ground again, $s = 0$; and, substituting this value of s in the equation $s = ut - \frac{1}{2}gt^2$, we have

$$0 = ut - \tfrac{1}{2}gt^2, \quad \text{or} \quad 0 = u - \tfrac{1}{2}gt; \therefore t = \frac{2u}{g},$$

which is the total time of ascent and descent, which is double the time of ascent; ∴ etc.

181. **To find the greatest height reached.**—When the greatest height is attained, $v = 0$; and, on substituting this value in the equation $v^2 = u^2 - 2gs$, we have

$$0 = u^2 - 2gs, \text{ which gives } s \text{ or } h = \frac{u^2}{2g},$$

where h is the greatest height.

182. By means of the equation $v^2 = u^2 - 2gs$, it can also be shown that the body will return to the ground with the same velocity as it had at starting, for, on returning to the Earth, $s = 0$;

$$\therefore \quad v^2 = u^2 - 2g \times 0, \quad \text{or} \quad v^2 = u^2, \quad \text{or} \quad v = u; \therefore \text{etc.}$$

Ex. 1. A body is projected vertically upwards with a velocity of 80 feet per second: find (1) the time taken to reach its highest point, (2) the greatest height reached, (3) its velocity at the end of 3 seconds:—

(1) $v = u - gt$, or $0 = 80 - 32t$; ∴ $t = 2\frac{1}{2}$ secs.

(2) $v^2 = u^2 - 2gs$, or $0 = 80^2 - 2 \times 32 \times s$; ∴ $s = 100$ feet;

(3) $v = u - gt$, or $v = 80 - 32 \times 3$, or $v = -16$.

This means that the velocity at the end of 3 seconds is 16 *f/s downwards*.

Ex. 2. A body is projected vertically downwards from the top of a cliff 300 feet high, and reaches the base in 4 seconds: find the velocity of projection.

Here $s = ut + \frac{1}{2}gt^2$, or $300 = u \,.\, 4 + \frac{1}{2}32 \times 4^2$, or $300 = 4u + 256$;

∴ $u = 11$ *f/s*.

Ex. 3. A body is projected vertically upwards with a velocity of 80 *f/s*: after what times will its height be 99 feet?

Here $s = ut - \frac{1}{2}gt^2$, or $99 = 80t - 16t^2$,

or $16t^2 - 80t + 99 = 0$, or $(4t - 9)(4t - 11) = 0$;

∴ $t = 2\frac{1}{4}$ or $2\frac{3}{4}$ seconds.

This means that at the end of $2\frac{1}{4}$ seconds it is at a height of 99 feet, and is ascending; and at the end of $2\frac{3}{4}$ seconds it is at the same height in its descent.

Ex. 4. When a balloon is at a height of 300 feet, a stone is dropped out and reaches the ground in 5 seconds: find the velocity of the balloon at the moment the stone was dropped.

The body in this case does not start from rest, but has initially the same velocity as the balloon. In fact, if, at the moment the body is dropped, the balloon has an upward velocity, the body first ascends, and, after remaining at rest for an instant, commences to descend.

Here $s = -300$ (the minus sign being given as it is measured *below* the point of projection)

$$s = ut - \tfrac{1}{2}gt^2, \text{ or } -300 = u \times 5 - 16 \times 5^2; \quad \therefore \quad u = 20\,f/s;$$

and, as this value is positive, the balloon must have been going *upwards* with a velocity of $20\,f/s$.

Ex. 5. A stone is let fall from the top of a precipice, and 1 second afterwards a second stone is projected vertically downwards from the same point with a velocity of 80 feet per second: find when and where the second stone will overtake the first.

At the moment the second stone is projected the first will have fallen a distance

$$s = \tfrac{1}{2}32 \times 1^2 = 16 \text{ feet.}$$

Also it will have a velocity $v = 32 \times 1 = 32$ feet per second.

Hence, at this moment, the velocities of the two stones are 32 and 80 feet per second respectively; and therefore the second stone is approaching the first with a *relative* velocity of $80 - 32$ or 48 feet per second. And, since both velocities are accelerated by the same amount, this relative velocity is constant throughout the motion. Hence, since their distance apart is 16 feet, and they approach one another at a uniform velocity of 48 feet per second, it follows that they will be together in $\frac{16}{48}$ or $\frac{1}{3}$ of a second after the projection of the second stone, or $1\frac{1}{3}$ seconds after the time when the first was dropped.

Also $s = \frac{1}{2}gt^2 = \frac{1}{2}32\,(1\frac{1}{3})^2 = 28\frac{4}{9}$ feet.

Hence, the stones will have fallen $28\frac{4}{9}$ feet when the second overtakes the first.

EXAMPLES XXV.

In the following examples the value of g is taken as $32 f/s/s$, or 981 $c/s/s$:—

1. A body is projected vertically upwards with a velocity of 160 feet per second: find (1) the time of ascent, (2) the total time of flight, (3) the greatest height reached.

2. With what velocity should a body be projected vertically upwards in order to ascend $30\frac{1}{4}$ feet? Express your answer both in feet per second and in miles per hour.

3. A body is let fall from rest under the action of gravity: find the space through which it falls (1) in 9 seconds, (2) in the 9th second after being let drop.

4. A body is projected vertically upwards with a velocity of 80 feet per second: after what times will its height be 96 feet?

5. With what velocity should a body be projected vertically upwards in order to ascend 121 feet?

6. Show that the height of a cliff in feet is equal to the square of the number of quarter seconds occupied by a body in falling from the top.

Here $s = \frac{1}{2}gt^2 = 16t^2 = (4t)^2$; but t being the number of seconds, $4t$ is the number of quarter seconds; $\therefore$ etc.

7. A body is projected vertically upwards with a velocity of 100 feet per second: after what times will its velocity be 36 feet per second (regarded as either upwards or downwards), and how high will it then be?

8. A man standing at the top of a cliff throws a stone vertically upwards with a velocity of 80 feet per second: if the stone reaches the bottom of the cliff after 6 seconds, find the height of the cliff.

9. With what velocity should a man, standing at the top of a cliff 224 feet high, project a stone vertically upwards in order that it may reach the foot of the cliff in 8 seconds?

10. A body let fall from rest falls through 80 feet in a particular second: find how long previous to the commencement of this second the body was in motion.

11. A man, descending the shaft of a mine with a velocity of 10 feet per second, drops a stone which reaches the bottom in 2 seconds: through what distance did it fall?

12. Two bodies are dropped successively from the same point with an interval of $\frac{1}{4}$ second: when will the distance between them be 8 feet?

13. A body is dropped through a height of 1024 feet: with what velocity must another be thrown down 3 seconds later in order that they may reach the ground together?

14. A body is thrown upwards from the top of a cliff with a velocity of 40 feet per second; 1 second afterwards another is thrown up from the same point with a velocity of 20 feet per second: after how many seconds will they meet, and where will they meet?

15. A body is dropped from the top of a tower 100 feet high, and at the same instant another is projected vertically upwards from the bottom with a velocity of 80 feet per second: when and where will they meet?

16. The value of the acceleration of gravity on Mars is half that which it is on the Earth: prove that the velocity acquired by a body in falling a given height on Mars is to that acquired in falling the same height on the Earth as

$$1 : \sqrt{2}.$$

17. At Cambridge a body falls through $80\frac{1}{2}$ feet in the 3rd second of its motion: find the value of g at Cambridge.

18. When a balloon is at a height of 500 feet, a stone is dropped out and reaches the ground in 5 seconds: find the velocity of the balloon at the moment the stone was dropped; also state whether the balloon was ascending or descending.

19. A balloon is ascending at the rate of 10 feet per second when a stone is dropped out which reaches the ground in 6 seconds: at what height was the balloon when the stone was let fall?

20. A balloon, on leaving the ground, ascends with a uniform velocity for 70 seconds when a stone is dropped out; after 10 seconds have elapsed the stone reaches the ground: find the velocity of the balloon, and its height when the stone was let fall.

21. A cricket-ball is thrown vertically upwards: find through what distance it goes in the last half second of its ascent.

22. A body at Paris falls from rest a distance of 14·715 metres in the 2nd second; find the acceleration of gravity at Paris.

23. A body is projected vertically upwards with a velocity of 981 centimetres per second: find how high it goes, and after what time it will reach the ground again?

24. A stone is dropped down a well 100 feet deep, and the sound of the splash is heard after an interval of $2\frac{33}{56}$ seconds: find the velocity of sound.

25. *A* and *B* are two points in a vertical line, *A* being 80 feet above *B*; a body is projected vertically downwards from *A* with a velocity of 40 feet per second, and at the same time another body dropped from *B*: find in what time the former body will overtake the latter.

26. There are two cages descending the shaft of a mine, one of which is falling freely under the action of gravity, and the other is descending with uniform velocity. A man in each cage releases a stone out of his hand. Find what will be the motion of the stone in each case.

27. A man throws a stone vertically upwards from the top of a shaft of a mine with a velocity of 160 feet per second. If he lets another stone drop down the shaft at the instant the first is within 20 feet of his hand on its return journey, at what distance below his hand will the two stones meet?

28. A stone is dropped down a well, and in $7\frac{7}{10}$ seconds the sound of the splash is heard. Find the depth of the well, the velocity of sound being 1120 feet per second.

CHAPTER XII.

LAWS OF MOTION.

183. **Kinematics and Kinetics.**—In the preceding Chapter we have considered the motion of a body in the abstract without reference to the causes which produce the motion: this branch of Dynamics is called *Kinematics.* We shall now inquire into the relation between the Force producing the motion and the Mass of the body moved. To this portion of Dynamics the term *Kinetics* is applied.

Newton's Laws of Motion.

Lex I.—"*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*"

Law I.—*Every body continues in its state of rest or of uniform motion in a straight line, unless compelled to alter that state by impressed force.*

Lex II.—"*Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*"

Law II.—*The change in the quantity of motion is proportional to the impressed force, and takes place in the direction of that force.*

Lex III.—"*Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*"

Law III.—*To every action there is an equal and opposite reaction: or the mutual actions of two bodies are always equal and act in opposite directions.*

184. *First Law.*—This law is sometimes called the Law of Inertia, for it states that no body of itself has any power to alter its condition of rest or of uniform motion in a straight line, and that therefore, if there is no force acting on a body, it must either be at rest or moving with uniform velocity in a straight line; and conversely, if a body be in a state of rest or moving uniformly in a straight line, there is no resultant force acting on the body.

But it might be said that a train moving with a uniform velocity of, say, 60 miles per hour, along straight level rails must require some force to keep up that speed; for, if the steam from which the energy is derived be shut off, the train will very soon be brought to rest; or, again, surely a cyclist when travelling along a straight level road is conscious of exerting a considerable muscular effort to maintain without increasing the speed at which he is travelling. This is quite true, but in each case the force which drives the train or bicycle forward is exactly equal and opposite to the sum-total of the forces which tend to retard it, such as friction and the resistance of the air, and therefore there is no resultant force. If, however, the forward force be greater than the combined retarding forces, the velocity of the train or cyclist becomes greater and greater; and, on the other hand, should the retarding forces exceed the driving force, there is a gradual diminution in the velocity.

Many illustrations of the First Law of Motion drawn from everyday experience will naturally occur to the student; for example, when a man jumps off a tram when in motion, his body continues to move with the same velocity as that of the tram; but his feet on touching the ground are suddenly brought to rest; he is therefore, owing to this velocity, flung on the ground.

185. *Second Law.*—This law enables us to measure a force.

Definition: The **Momentum** or **Quantity of Motion** of a body is measured by the *product of its mass and velocity.*

Thus, if m denotes the mass of a body, and v its velocity, the momentum $= mv$.

The *unit of momentum* is that of *the unit of mass moving with unit velocity*, which, when expressed in F. P. S. units, means the momentum possessed by a mass of one pound when moving with a velocity of one foot per second. Thus a mass of 10 lbs. moving with a velocity of 6 feet per second possesses 10×6 or 60 units of momentum.

186. We thus see that by "change in the quantity of motion" in Newton's Second Law is meant "change in momentum"; but in estimating the change in momentum we must obviously take the idea of TIME into consideration, for a force acting on a certain mass for two seconds will produce twice the velocity, and therefore twice the momentum, which the same force would produce in the same mass if it acted for one second only. The Law will therefore become more definite if "rate of change of momentum" (i.e. the change per unit of time) be substituted instead of "change in the quantity of motion." The Law will now be as follows:—

The rate of change of momentum is proportional to the impressed force and takes place in the direction of that force.

187. **Measurement of Force.**—Newton's First Law supplies us with the definition of Force given in Art. 7; but by means of the Second Law a Force may be measured thus:—

Let P be the measure of a force which acts on a body whose mass is m, and f the acceleration produced in the body by the action of the force; then by the Second Law P is proportional to the rate of change of momentum, i.e. to the rate of change of mv. But the rate of change of v is the acceleration f; hence the rate of change in mv is mf: therefore P is proportional to mf. Therefore $P = kmf$ where k is some constant.

Now, if we take the unit of force to be *that force which produces unit acceleration in unit mass*, then, when $m = 1$, and $f = 1$, P will $= 1$;

$$\therefore\ 1 = k \times 1 \times 1;\quad \therefore\ k = 1;\quad \therefore\ P = mf.$$

Hence, if the above units are chosen, it follows that *the number which measures the force is equal to the number which measures the rate of change of momentum.*

188. **The Poundal.**—When the unit of mass is 1 lb., and the unit of acceleration that of 1 foot per sec. per sec., the corresponding unit of force is called a poundal, that is:—

The Poundal is that force which, acting on a mass of one pound, produces in it an acceleration of one foot per sec. per sec.

189. The Poundal is an *absolute unit* of force, by which is meant that this unit so defined is the same at all parts of the Earth, and indeed at all parts of the universe, and does not depend on locality, as is the case with gravitation-units.

190. **The Dyne.**—The absolute unit of force in the C. G. S. system is called the Dyne, which may therefore be defined as that force which, acting on a mass of one gramme, produces in it an acceleration of one centimetre per sec. per sec.

191. *Connexion between the Poundal and the weight of a Pound.*—If we take a mass of one pound and let it fall freely under the action of gravity, it will move with an acceleration of g (Art. 176); hence, a force of one pound weight, acting on a mass of one pound, produces in it an acceleration of $g\,f/s/s$; but—

1 Poundal produces in 1 lb. mass an acceleration of $1\,f/s/s$. Therefore a weight of one pound must contain g poundals, or

$$1 \text{ poundal} = \frac{1}{g} \text{ lb. wt.}$$

Taking g as 32 units, we see that approximately

$$\text{a poundal} = \frac{1}{32} \text{ lb. wt.} = \frac{1}{2} \text{ oz. wt.}$$

Similarly in the C. G. S. system where $g = 981\ c/s/s$ the approximate value of a Dyne is $\frac{1}{981}$ of a gramme wt.

192. The student should bear in mind that if m be the mass of the body expressed in pounds, and f the acceleration, expressed in foot-second units, produced in it by the action of P, then $P = mf$ determines the value of P in *poundals* from which the value in gravitation-units, i.e. pounds weight, can be obtained by dividing by g. The same rule also applies when the units are in the C. G. S. system.

Ex. 1. A force of 160 poundals acts on a mass of 40 pounds: find the acceleration produced.

Here $P = mf$ or $160 = 40f$; $\therefore f = 4f/s/s$.

Note.—Observe that the force is expressed in poundals and the mass in pounds.

Ex. 2. A force equal to the weight of 1 ounce acts on a body whose mass is 1 pound: find how far it moves it in 1 minute.

Here $P = 1$ ounce $= \frac{1}{16}$ lb. $= \frac{1}{16} \times 32 = 2$ poundals.

Now $P = mf$ or $2 = 1 \times f$; $\therefore f = 2f/s/s$, and distance

$$s = \tfrac{1}{2}ft^2 = \tfrac{1}{2} \times 2 \times (60)^2 = 3600 \text{ feet}.$$

Ex. 3. A constant force acting on a mass of 1 ton moves it over a distance of 14 feet in 14 seconds: find the magnitude of the force.

Here $14 = \frac{1}{2}f\,14^2$; $\therefore f = \frac{1}{7}$, and

$$P = mf = 2240 \times \tfrac{1}{7} = 320 \text{ poundals} = 10 \text{ lbs. wt.}$$

Ex. 4. A mass of 20 lbs. is moving on a smooth horizontal table with a velocity of 8 feet per second: (*a*) what force would stop it in 2 seconds? (*b*) also if the same mass were falling vertically under the action of gravity with a velocity at a particular instant of 8 feet per second, what force applied at that instant would stop it in 2 seconds?

(*a*) Here the acceleration $f = -4$ ft.-sec. units, and the

$$\text{force} = mf = 20 \times 4 = 80 \text{ poundals} = 2\tfrac{1}{2} \text{ lbs. wt.}$$

(*b*) Here the body has an acceleration of g, or $32f/s/s$ downwards, and the force changes this to a retardation of $4f/s/s$; hence, the force actually produces a total retardation of $36f/s/s$, and

$$P = mf, \text{ or } P = 20 \times 36 \text{ poundals} = 22\tfrac{1}{2} \text{ lbs. wt.}$$

EXAMPLES XXVI.

1. A force of 80 poundals acts on a mass of 10 pounds: find (*a*) the acceleration produced; (*b*) the time taken to get up a velocity of 60 miles per hour; (*c*) the space described in that time, the body being supposed to start from rest.

2. A mass of 1 ton on a smooth horizontal plane is acted on by a force equal to a weight of 5 pounds acting parallel to the plane: find the space through which it is moved in 7 seconds.

3. A body whose mass is 48 pounds, starting from rest under the action of a constant force, passes over 300 yards in 10 seconds: find the force (*a*) in absolute units, (*b*) in gravitation units.

4. Find the magnitude of the uniform force which would generate in 1 second a velocity of 8 feet per second in a mass of 6 pounds.

5. A body resting on a smooth horizontal plane is acted on by a horizontal uniform force of 1 lb. wt., and moves over a distance of 10 feet in 4 seconds: find the mass of the body.

6. A constant uniform force of 10 lbs. wt., acting on a body originally at rest, produces in it a velocity of 1 foot per second at the end of 7 seconds: find the mass of the body.

7. How long must a uniform force of $1\frac{1}{2}$ ozs. wt. act on a mass of 3 lbs. in order to produce in it a velocity of 40 feet per second?

8. How long must a uniform force of 5 lbs. wt. act on a mass of 20 lbs. in order that, starting from rest, it may move over a space of 100 feet? Find also the momentum of the body at the end of that time.

9. What force must act on a truck whose mass is 20 tons, moving on smooth horizontal rails, in order to increase its velocity from 30 miles per hour to 60 miles per hour while it moves over 121 yards?

10. A body whose mass is 20 lbs. is thrown along the surface of a frozen lake with a velocity of 80 feet per second, and comes to rest in 20 seconds: find the retarding force.

11. A train is travelling along horizontal rails with a velocity of 30 miles per hour when steam is shut off, and the train comes to rest in 88 seconds: if the mass of the train be 200 tons, find the magnitude of the retarding force.

12. A train is travelling along horizontal rails with a velocity of 15 miles per hour when steam is shut off; if the resistance due to friction, etc., be 10 lbs. wt. for each ton of the mass of the train, find how far it will travel before coming to rest.

13. (*a*) What constant force acting on a mass of 20 lbs. will move it along a smooth horizontal plane through a distance of 50 feet in 5 seconds? (*b*) What force would lift the same mass through a vertical height of 50 feet in 5 seconds, the body in each case starting from rest?

14. What constant force acting on a mass of 1 ton resting on a smooth horizontal plane will produce in it a velocity of 8 feet per second when it has moved it over a distance of 16 feet? Also, find what force acting vertically upwards on the same mass would produce the same velocity as before when it has raised it through the same distance, the body in each case starting from rest.

15. A train moving on horizontal smooth rails increases its speed uniformly from starting: if it takes 2 minutes to describe the first mile, compare the force exerted by the engine with the total weight of the train (including the engine).

16. A force acting on a certain mass produces in it a velocity of 12 feet per second at the end of 3 seconds: if twice this force acted on a body of half the mass, how far would it move the body in 2 seconds?

17. A mass of 8 ounces is moving along a smooth horizontal plane with a velocity of 20 feet per second: what force will stop it in 25 feet?

18. In Example 17, if the same mass be falling vertically, what force applied when its velocity is 20 feet per second will stop it in 25 feet?

19. A mass of 20 pounds is acted on during a period of 6 seconds by a constant force of 5 lbs. wt.: the force then ceases to act, and the body continues its motion for 2 seconds, during which time no force acts on it; it is then subjected to a retarding force of 10 lbs. wt. until the body is brought to rest: find the total distance traversed.

20. A constant force equal to the weight of one gramme acts on a mass of one kilogramme: find the space described in one minute.

21. What constant force acting vertically upwards will raise a mass of one pound through one foot in one second?

22. A mass of one pound rests on a horizontal plane: what horizontal force will (*a*) move the body through a distance of one foot in one second? (*b*) will produce a velocity of one foot per second at the end of one second?

23. What constant horizontal force will (*a*) move a mass of one gramme placed on a horizontal smooth table through a distance of one centimetre in one second? (*b*) produce in it a velocity of one centimetre per second at the end of one second, the body being supposed to start from rest?

The Weights of different Bodies are proportional to their Masses.

193. We have seen (Art. 174) that all bodies fall with the same acceleration, viz. g; and, since the weight W of a body acting on its mass m produces this acceleration, the equation $P = mf$ becomes $W = mg$.

Therefore, if m_1 and m_2 represent the masses of any two bodies whose weights are W_1 and W_2, we have

$$W_1 = m_1 g, \quad \text{and} \quad W_2 = m_2 g;$$

$$\therefore \quad W_1 : W_2 :: m_1 : m_2; \ \therefore \text{\&c.}$$

Hence, if the weights of two bodies when compared by an ordinary balance are found to be equal, their masses are also equal; and therefore the weight of a body is a measure of its mass.

It is to be borne in mind that, in the equation $W = mg$, W represents the number of absolute units of force in the weight of the body; thus, in the F.P.S. system, the number of poundals in W = the number of pounds in $m \times g$.

194. **Difference between Mass and Weight.**—The student should always remember that the *mass of a body is the quantity of matter contained in the body,* and is therefore independent of the position of the body on the Earth, or, indeed, anywhere in the universe; whereas the *weight of a body is the force with which it is attracted to the centre of the Earth,* and is therefore a variable quantity depending on the acceleration of gravity at the particular part of the Earth where the body happens to be situated. Thus at the Equator, where $g = 32{\cdot}09 f/s/s$, a mass of 10 lbs. would weigh $32{\cdot}09 \times 10$ or 320·9 poundals; whereas at London, where $g = 32{\cdot}19$, the weight of a 10-lb. mass would be

$$32{\cdot}19 \times 10 = 321{\cdot}9 \text{ poundals.}$$

195. An ordinary balance, such as is generally used for weighing, enables us to measure the masses of different substances; thus, if at London a body in one scale of a balance exactly balances a standard pound mass of metal in the other scale, it follows that the mass of the body is the same as that of the standard mass, viz. one pound. Also it is evident that, if brought to the North Pole, both the body and the standard pound mass would weigh slightly more than in London; but the increase in each case would be the same, and they would therefore still balance when placed in the scales.

196. A spring-balance is also often used for weighing; in this case, the body is suspended from the extremity of a spiral spring which, being extended by the weight of the

body, moves an index-hand which travels along a graduated scale. In this case, what is actually measured is not the mass of the body, but the force which extended the spring, or, in other words, the weight of the body; and therefore, if a certain mass when weighed in London indicated one pound weight on the graduated scale, a slightly smaller mass would, at the North Pole, have to be attached to the balance to indicate the same reading of 1 lb. wt.; and thus the force in each case is the same, but not the mass.

197. **Deductions from Newton's Second Law.**— The following important facts are implied in this law:—

(1) *A force will produce the same change in the quantity of motion of a body, whether the body be at rest or in motion at the moment the force is applied.*

(2) *When a body is acted on simultaneously by a number of forces, each force will produce the same change in the quantity of motion as if it acted alone on the body.*

This latter statement is frequently referred to as the Principle of the "**Physical Independence of Forces.**"

Instances of the first of these principles are matters of everyday experience; for example, if a ball be rolled along the deck of a ship, it will go the same distance whether the ship be at rest or in motion. In fact, when people play any game on board a ship, such as cricket or quoits, the game is played with exactly the same strokes and pitches when the ship is in motion as would be made if it were at rest.

Again, if a stone be projected horizontally from the top of a cliff or tower, and at the same instant another stone be let drop from the same point, both will be found to reach the ground in the same time, the vertical acceleration of gravity thus moving both bodies through the same vertical distance in the same time.

198. From the very nature of the laws of motion, no *a priori* proof can be given of their truth; they have, however,

been tested by many accurate experiments, and deductions based on them are always verified by observed facts. Astronomy furnishes many striking examples which verify their truth ; for example, the exact instant at which an eclipse of the Sun or Moon or a transit of Venus will take place can be predicted by intricate calculations based on these laws; and it is found that the event takes place at the exact moment which has been foretold years beforehand. The discovery of the planet Neptune, which was one of the most brilliant in the history of Astronomy, would alone afford a strong presumption of the truth of Newton's Laws. "It was found that the positions which it was calculated the planet Uranus should occupy, after making allowance for all known disturbing forces, did not coincide with the observed positions. It was therefore thought that there must be some unknown planet whose attraction produced these disturbances. After the most laborious calculations, the position which this unknown body should occupy was determined at almost the same time by Leverrier in France, and Adams in England in the year 1846." *

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* "Elements of Astronomy," chap. vi.

CHAPTER XIII.

APPLICATION OF THE LAWS OF MOTION.

Motion of a particle on a smooth inclined plane.

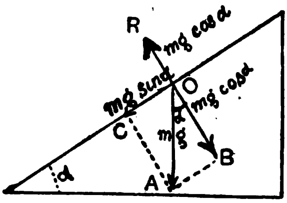


FIG. 90.

199. Let O be the position of the body on a smooth inclined plane whose inclination to the horizon is α. Lay down OA vertically downwards to represent mg, the weight of the particle. Then erect a perpendicular OB to the plane and complete the parallelogram OA. Therefore (Art. 32) mg resolves into two forces represented by OB and OC, i.e.,

$$mg\cos\alpha \text{ perpendicular to the plane,}$$

and $$mg\sin\alpha \text{ along the plane.}$$

But $mg\cos\alpha$ is neutralized by the reaction OR of the plane. Hence there remains the force $mg\sin\alpha$ down the plane, but

$$P = mf, \quad \text{or} \quad mg\sin\alpha = mf, \quad \therefore \quad f = g\sin\alpha.$$

Ex. 1. A particle is placed on a smooth inclined plane of inclination 30°: find (a) the acceleration with which it slides down, (b) the time taken to slide down the plane a distance of 72 feet.

Here the weight of the body mg resolves itself into $mg\sin 30$ along the plane, and $mg\cos 30$ perpendicular to the plane, the latter force being balanced by the reaction of the plane: therefore $mg\sin 30$ remains. But

$$P = mf \quad \text{or} \quad mg\sin 30 = mf,$$

$$\therefore \quad f = g\sin 30 = 32 \times \tfrac{1}{2} = 16 f/s/s;$$

also the space s or $72 = \tfrac{1}{2}ft^2 = \tfrac{1}{2}16t^2$,

$$\therefore \quad t^2 = 9, \quad \therefore \quad t = 3 \text{ seconds}.$$

Ex. 2. The inclination of a smooth plane to the horizon is $\sin^{-1}\frac{5}{13}$ (i.e. the height is to the length as 5 : 13), and a body is projected up the plane with a velocity of 80 feet per second: find (*a*) the distance gone by the body up the plane before the motion is reversed, (*b*) the distance gone by the body in the 7th second after projection.

(*a*) Here, as before, $f = -g \sin \alpha = -\frac{32 \times 5}{13}$,

$$v^2 = u^2 + 2fs,$$

or

$$0 = 80^2 - 2 \times \frac{32 \times 5}{13} s,$$

$$\therefore \quad s = \frac{80 \times 80 \times 13}{2 \times 32 \times 5} = 260 \text{ feet.}$$

(*b*) Space in 7th sec. = space in 7 secs. − space in 6 secs.

$$= \left\{ 80 \times 7 - \tfrac{1}{2}\,\frac{32 \times 5}{13} \times 49 \right\} - \left\{ 80 \times 6 - \tfrac{1}{2}\,\frac{32 \times 5}{13} \times 36 \right\} = 0,$$

which means that during the 7th second the body went up the same distance as it came down.

200. *The velocity acquired by a body in sliding down a smooth inclined plane is equal to that acquired in falling down the height of the plane.*

Let h and l represent the height and length respectively, and α the angle of inclination.

Now, generally, $v^2 = 2fs$; but in sliding down the plane

$$f = g \sin \alpha \quad \text{and} \quad s = l,$$

$$\therefore \quad v^2 = 2g \sin \alpha \,.\, l, \quad \text{but} \quad \sin \alpha = \frac{h}{l},$$

$$\therefore \quad v^2 = 2g\frac{h}{l} \,.\, l = 2gh, \quad \text{or} \quad v = \sqrt{2gh}.$$

Again, in falling down the height, $s = h$ and $f = g$,

$$\therefore \quad v^2 = 2fs = 2gh, \quad \text{or} \quad v = \sqrt{2gh}.$$

the same as before; ∴ &c.

201. *The time occupied by a body in falling down all chords of a vertical circle drawn from the highest or lowest point is constant, and is equal to the time occupied in falling through the vertical diameter.*

Let OA be a chord drawn through O, the highest point of a vertical circle, and let its length be l; also let d be the length of the vertical diameter OB.

Drop AX perpendicular on OB; and let

$$\angle OAX \quad \text{or} \quad \angle ABX = \alpha.$$

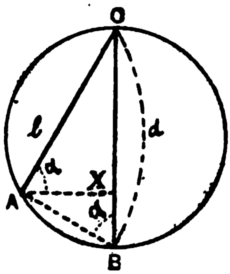


FIG. 91.

Now, $s = \frac{1}{2}ft^2$, but $s = l$ and $f = g \sin \alpha$,

$$\therefore \quad l = \tfrac{1}{2}g \sin \alpha t^2, \quad \therefore \quad t^2 = \frac{2l}{g \sin \alpha};$$

but $\sin \alpha = \frac{l}{d}$, $\therefore \quad t^2 = \dfrac{2l}{\frac{gl}{d}} = \dfrac{2d}{g}, \quad \therefore \quad t = \sqrt{\dfrac{2d}{g}}$;

but in the case of the vertical diameter

$$s = \tfrac{1}{2}ft^2, \quad \text{or} \quad d = \tfrac{1}{2}gt^2, \quad \therefore \quad t = \sqrt{\frac{2d}{g}},$$

which is the same as for the chord; ∴ &c.: similarly for the chord AB.

Motion on a rough inclined plane.

202. **Case I.**—When the body is moving down the plane. As before, lay down OA to represent mg, the weight of

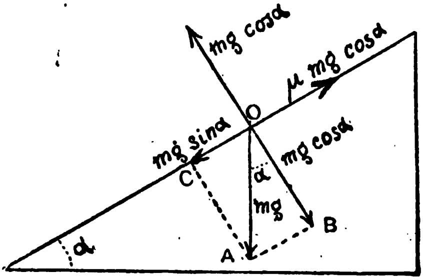


FIG. 92.

the body; this force resolves, as before, into $mg \sin \alpha$ along the plane, and $mg \cos \alpha$ perpendicular to the plane, the latter force being balanced by the normal reaction of the plane. There is also the force of friction acting up the plane, which $= \mu \times$ normal pressure $= \mu mg \cos a$.

Hence, the force which causes motion

$$= mg \sin a - \mu mg \cos a = mg(\sin a - \mu \cos a);$$

but $\qquad P = mf, \quad \therefore \; mg(\sin a - \mu \cos a) = mf,$

or $$f = g(\sin a - \mu \cos a).$$

203. **Case II.**—When the body is moving up the plane.

In this case $mg \sin a$ and $\mu mg \cos a$ both act down, and f is now a retardation, or, in other words, is negative;

$$\therefore \; f = -g(\sin a + \mu \cos a).$$

Ex. 1. A body is in motion along a rough inclined plane whose inclination is 30°, the coefficient of friction being $\frac{1}{4}$: find (*a*) the acceleration when moving down the plane, (*b*) the retardation experienced when projected up the plane.

(*a*) Here, as before, the force producing the motion is $mg(\sin 30 - \mu \cos 30)$, which must $= mf$;

$$\therefore \; f = g(\sin 30 - \mu \cos 30) = 32\left(\frac{1}{2} - \frac{\sqrt{3}}{8}\right) = 4(4 - \sqrt{3})\,f/s/s.$$

(*b*) In this case

$$f = -g(\sin 30 + \mu \cos 30) = -4(4 + \sqrt{3})\,f/s/s.$$

Ex. 2. A particle is projected up a rough inclined plane, whose inclination is $\sin^{-1}\frac{5}{13}$, with a velocity of 60 miles per hour; given the coefficient of friction to be $\frac{1}{2}$, how far up will it go, and what will it do then?

Here we see, on resolving along the plane, that the force which produces the retardation $= mg(\sin a + \mu \cos a)$, which must $= mf$;

$$\therefore \; f = -g(\sin a + \mu \cos a),$$

the negative sign being placed before it, as it is a retardation.

$$\therefore \; f = -32\left(\frac{5}{13} + \frac{1}{2}\,\frac{12}{13}\right) = -\frac{32 \times 11}{13};$$

but $v^2 = u^2 + 2fs$ or $0 = 88^2 - \frac{64 \times 11}{13}s$; $\therefore$ $s = 143$ feet.

Again, when the body reaches its highest point, it is pulled down the plane by a force

$$mg \sin a = \frac{mg5}{13},$$

and if in limiting equilibrium it would be pulled up the plane by a force

$$\mu mg \cos a = \frac{1}{2}mg\,\frac{12}{13} = \frac{mg6}{13}.$$

Since the upward force of friction would be greater than the resolved part of its weight downwards, the body will therefore remain at rest.

Motion on a rough horizontal plane.

204. Here the forces acting on the body are its weight, which $= mg$.

Also the normal pressure R, which $= mg$ (Third Law), and the friction which $= \mu \times$ normal pressure $= \mu mg$.

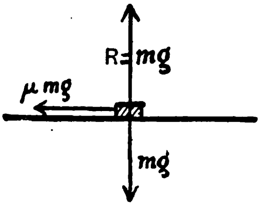


FIG. 93.

The normal pressure and the weight balance one another, and the only force causing change of motion is μmg; but

$$P = mf, \text{ or } \mu mg = mf; \quad \therefore \quad f = \mu g;$$

and as it is always a retardation, we write it

$$f = -\mu g.$$

Ex. 1. A stone is thrown along a frozen lake with a velocity of 20 feet per second; if the coefficient of friction be $\frac{1}{4}$, find how far it will travel, and what will be the interval of time before it comes to rest?

Here the retarding force $= \mu mg$, which must $= mf$;

$$\therefore \quad f = -\mu g = -\tfrac{1}{4} \times 32 = -8;$$

$$v^2 = u^2 + 2fs, \quad \text{or} \quad 0 = 20^2 - 16s; \quad \therefore \ s = 25 \text{ feet}.$$

Again, $\quad v = u + ft; \quad \therefore \ 0 = 20 - 8t; \quad \therefore \ t = 2\tfrac{1}{2}$ seconds.

EXAMPLES XXVII.

1. A smooth inclined plane has a rise of 1 in 4 (i.e. the height is to the length as 1 to 4): find the acceleration with which a particle will slide down it.

2. A particle starting from rest slides down a smooth inclined plane of length 24 feet in $\sqrt{3}$ seconds: find the inclination of the plane.

3. A particle starting from rest slides down a smooth inclined plane whose inclination is $\sin^{-1}\frac{5}{13}$, and whose length is 65 feet: find the velocity when it reaches the bottom.

4. A body is projected up a smooth incline whose angle of inclination is 30° with a velocity of 40 feet per second: find how far it will travel up the plane, and also the time which will elapse before it returns to the point from which it was projected.

5. A smooth inclined plane rises 1 in 8, and a body is projected up the plane with a velocity of 30 miles per hour: find how far it will go in the 12th second; also, if the result is negative, interpret the meaning of the sign.

6. The height of a smooth inclined plane is 100 feet, and its length 260 feet: with what velocity should a body be projected up the plane in order just to reach the top? Also, if when the body reaches the top, it topples over and falls vertically down the height, find the velocity it will have acquired when it reaches the bottom. Is it really necessary that the length of the plane should be given to solve this question, and if not, why not?

7. A number of bodies are simultaneously let slide from rest from the tops of a number of smooth inclined planes of different inclinations and different lengths: prove that, if the heights of all the planes are the same, the bodies will all reach the bottom with the same velocity.

8. A smooth inclined plane is 18 feet long: divide it into three parts, so that a body starting from rest at the top may pass over each of the parts in the same time.

Note.—It is evident, from Art. 173, that the lengths of the three parts must be proportional to the odd numbers 1, 3, 5.

9. Find the constant force necessary to increase the speed of a train of mass 200 tons from 30 miles per hour to 60 miles per hour in 10 minutes moving up an incline of 1 in 100, the resistance due to friction being 10 lbs. wt. per ton.

Here the resolved part of the weight along the plane = $mg \sin \alpha$ poundals. Hence, if P poundals be the required force, we have

$$P - mg \sin \alpha - \text{Friction} = mf,$$

but the friction = 10 × 200 lbs. wt. = 10 × 200 × 32 poundals;

$$\therefore \quad P - 200 \times 2240 \times 32 \times \frac{1}{100} - 10 \times 200 \times 32 = 200 \times 2240 \times f.$$

Also, the acceleration is thus found:—

$$v = u + ft, \quad \text{or} \quad 88 = 44 + 600f; \quad \therefore f = \frac{44}{600} = \frac{11}{150};$$

substituting this value of f we obtain

$$P = 7506\tfrac{2}{3} \text{ lbs. wt.}$$

10. Find the constant force necessary to move a train of mass 150 tons up an incline of 1 in 150 through a distance of half a mile in one minute, starting from rest, the resistance due to friction being 12 lbs. wt. per ton.

11. Find the force necessary to move a train of mass 200 tons up an incline of 1 in 100 with uniform velocity, the resistance due to friction being 15 lbs. wt. per ton.

12. A body is projected up a rough inclined plane whose inclination is $\sin^{-1}\frac{3}{5}$ with a velocity of 24 feet per second; if the coefficient of friction be $\frac{1}{4}$, how far will it ascend? After reaching its highest point, what will the body then do? Also, if it is found that the body slides down the plane again, with what velocity does it return to the point from which it was projected?

13. A mass of 20 tons is sliding down a smooth incline of 1 in 80: find what constant force applied at the moment the velocity of the body is 60 miles per hour will stop it (*a*) in 200 yards, (*b*) in 2 minutes.

14. A body is thrown along a horizontal plane with a velocity of 40 feet per second: if the coefficient of friction be ·5, find (*a*) how far, and (*b*) how long, it travels before coming to rest.

15. A sledge is given an initial velocity of 30 miles per hour along the surface of a frozen lake, and travels a distance of 24 feet in the third second after starting: find the retardation and the coefficient of friction.

16. Find the force necessary to move a train of mass 150 tons on horizontal rails through a distance of 200 yards in 20 seconds, starting from rest, the resistance due to friction being 10 lbs. wt. per ton.

17. Find the line of quickest descent to the circumference of a vertical circle from a point outside the circle situated in the plane of the circle.

Pressure of a body on a horizontal plane which is either ascending or descending with an acceleration.

205. **Case I.**—Let the plane AB ascend with an acceleration f, and let R be the pressure between a body whose mass is m and the plane.

The only forces acting on the body are the reaction of the plane R upwards, and the weight of the body mg downwards; and since the acceleration is vertically upwards, ∴ R must be greater than mg, and the resultant force $= R - mg$.

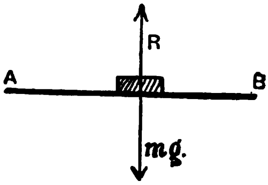


FIG. 94.

This is the force which produces the acceleration of f in the body whose mass is m. But $P = mf$; ∴ $R - mg = mf$, from which R may be found when m and f are given.

206. **Case II.**—Again, let the plane and body descend with an acceleration f; then, as the acceleration is downwards, the downward force mg must be greater than R, and the resultant force acting on the body is $mg - R$; but $P = mf$; ∴ we have $mg - R = mf$, from which R may be found as before.

Corollary.—If the plane be either ascending or descending with *uniform velocity*, then $f = 0$; and from either of the above equations we infer that $R = mg$, that is, the pressure on the plane is equal to the weight of the body, and is, therefore, the same as if the plane were at rest. This result also follows without the aid of the above formulæ directly from Newton's Second Law; for, according to this law, when the body is at rest or in uniform motion in a straight line, there can be no resultant force acting on the body;

$$\therefore \quad R = mg = \text{the weight of the body.}$$

Ex. 1. A body of mass 20 lbs. is on a lift; find the pressure of the body on the lift when—

(*a*) the lift is at rest;
(*b*) the lift is ascending with a uniform velocity of $4 f/s$;
(*c*) the lift is descending with a uniform velocity of $4 f/s$;
(*d*) the lift is ascending with a uniform acceleration of $4 f/s/s$;
(*e*) the lift is descending with a uniform acceleration of $4 f/s/s$.

Evidently, in each of the cases (*a*), (*b*), and (*c*), the pressure is equal to the weight of the body, viz. 20 lbs. wt.

In (*d*) we have $R - mg = mf$, or $R - 20 \times 32 = 20 \times 4$;

$$\therefore \quad R = 720 \text{ poundals} = 22\tfrac{1}{2} \text{ lbs. wt.}$$

In (*e*) we have $mg - R = mf$, or $20 \times 32 - R = 20 \times 4$;

$$\therefore \quad R = 560 \text{ poundals} = 17\tfrac{1}{2} \text{ lbs. wt.}$$

Ex. 2. A mass of half a pound is hung on to a spring-balance in a balloon: find the acceleration with which the balloon is moving (*a*) when the balance indicates 9 ozs., (*b*) when it indicates 7 ozs.

(*a*) Here R is 9 ozs. wt. $= \dfrac{9}{16} \times 32 = 18$ poundals. In this case, as R is greater than the weight, the acceleration is upwards.

$$R - mg = mf,$$

$$18 - \tfrac{1}{2}32 = \tfrac{1}{2}f; \quad \therefore \; f = 4 f/s/s \text{ upwards.}$$

(*b*) Here R is 7 ozs. wt. $= \dfrac{7}{16} \times 32 = 14$ poundals, and acceleration is downwards.

$$mg - R = mf,$$

$$\tfrac{1}{2}32 - 14 = \tfrac{1}{2}f; \quad \therefore \; f = 4 f/s/s \text{ downwards.}$$

Note.—It is evident, from the above equations or otherwise, that if a body be placed on a horizontal plane and both fall freely under the action of gravity, i.e. both fall with an acceleration of g, during the fall the pressure between the body and the plane will be zero; or, in other words, the body during the fall will barely be in contact with the plane, and there will be no reaction between them.

EXAMPLES XXVIII.

1. A mass of 40 lbs. hangs by a rope from a balloon: find the tension of the rope when—

(*a*) the balloon is at rest;
(*b*) the balloon is ascending with a uniform velocity of 8*f*/*s*;
(*c*) the balloon is descending with a uniform velocity of 8*f*/*s*;
(*d*) the balloon is ascending with a uniform acceleration of 8*f*/*s*/*s*;
(*e*) the balloon is descending with a uniform acceleration of 8*f*/*s*/*s*;
(*f*) the balloon is ascending with a uniform acceleration of g;
(*g*) the balloon is descending with a uniform acceleration of g.

2. A mass of 1 cwt. is on a lift which is (*a*) ascending with a velocity which is increasing at the rate of 4 foot-second units, (*b*) ascending with a velocity which is decreasing at the rate of 4 foot-second units: find in each case the pressure on the floor of the lift.

3. A mass of 1 cwt. is hung on a spring-balance in a balloon: find what the index of the balance will indicate under the following circumstances:—(*a*) when the balloon is descending with a velocity which is increasing at the rate of 4 foot-second units; (*b*) when it is descending with a velocity which is decreasing at the rate of 4 foot-second units.

4. If a person, holding a mass of 10 lbs. on his hands, jump off a table, find the pressure of the body on his hand as he falls in the air.

5. If I hold a body on my hand, with what acceleration should I move my hand vertically in order that (*a*) the pressure on my hand should be zero? (*b*) the pressure on my hand should be twice the weight of the body?

6. A mass of 100 lbs. is on the floor of a lift: what is the nature of the motion of the lift when the pressure of the body on the floor is (*a*) 100 lbs. wt., (*b*) 150 lbs. wt.

7. A mass of 1 lb. is on my hand; the pressure on my hand is 15 ozs. wt.: with what vertical acceleration is my hand in motion?

CHAPTER XIV.

FURTHER APPLICATIONS OF THE LAWS OF MOTION.

207. **Atwood's Machine.** *Motion of two unequal masses connected by a string which passes over a smooth pulley.*

Atwood's machine consists essentially of two unequal masses m and m' connected by an inextensible string which passes over a smooth pulley A. A more complete description of this machine, and the uses to which it is applied, will be given towards the end of this chapter. We have now to consider the nature of the motion.

Let m be greater than m'; then, since the string is inextensible, the velocity of m at any instant must be the same as that of m'; and, hence, the accelerations of both bodies are the same. Let this common acceleration be f. Also the tension T of the string is the same throughout its length.

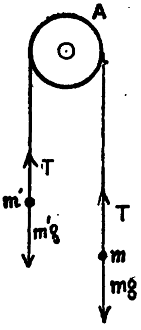


Fig 95.

Now, the mass m is acted on by its own weight mg downwards, and T upwards, mg being $> T$; therefore the resultant force is $mg - T$ poundals;

$$\therefore \quad mg - T = mf. \tag{1}$$

Similarly the forces acting on m' are T upwards, and $m'g$ downwards, T being $> m'g$;

$$\therefore \quad T - m'g = m'f; \tag{2}$$

adding these two equations we have

$$(m - m')g = (m + m')f; \quad \therefore \quad f = \frac{m - m'}{m + m'} g.$$

Hence the acceleration f is found.

Also, to find T substitute this value of f in (2), and we have

$$T = m'g + \frac{(m - m')m'}{m + m'}g = \frac{2mm'}{m + m'}g \text{ poundals.}$$

Note.—In working numerical examples the student is advised not to make use of these formulae, but to apply first principles directly to each example.

Ex. 1. Two masses of 9 and 7 lbs. are connected by a string passing over a smooth pulley: find the distance through which each moves in 3 seconds. Also find the tension of the string.

Here $9g - T = 9f$ (1)

and $T - 7g = 7f$ (2)

add $\therefore$ $2g = 16f$; $\therefore$ $f = 4f/s/s$;

also $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 4 \times (3)^2 = 18$ feet.

Again, to find T substitute the value of f in equation (2);

$\therefore$ $T - 7 \times 32 = 7 \times 4$; $\therefore$ $T = 252$ poundals $= 7\frac{7}{8}$ lbs. wt.

Ex. 2. Two masses of 5 and 3 lbs. are hung over a smooth pulley Motion now continues for 5 seconds when the string breaks: find through what distance the lighter will continue to ascend before its motion is reversed.

Here $5g - T = 5f$

also $T - 3g = 3f$

$\therefore$ $2g = 8f$; $\therefore$ $f = 8f/s/s$.

Again, at the end of 5 seconds the velocity $= 8 \times 5 = 40f/s$. Now, when the string breaks, the lighter mass has an *initial* upward velocity of $40f/s$, and the acceleration acting on it now is that due to gravity, viz., $-32f/s/s$;

$\therefore$ $v^2 = u^2 + 2fs$ or $0 = 40^2 - 64s$; $\therefore$ $s = 25$ feet.

Ex. 3. Two strings pass over a smooth pulley; at the extremities of the strings on one side are hung masses of 5 and 4 lbs. respectively, and on the other side both strings are attached to a mass of 7 lbs.: find the acceleration and the tensions of the strings.

Here we write down the equation of motion of each mass separately, thus—

$5g - T_1 = 5f$ (1)

$4g - T_2 = 4f$ (2)

$T_1 + T_2 - 7g = 7f$ (3)

add $2g = 16f$; $\therefore$ $f = 4f/s/s$.

Substituting this value of f in (1), we have

$5 \times 32 - T_1 = 5 \times 4$; $\therefore$ $T_1 = 140$ poundals $= 4\frac{3}{8}$ lbs. wt.;

and similarly $T_2 = 3\frac{1}{2}$ lbs. wt.

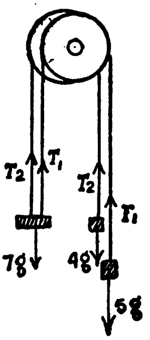


FIG. 96.

Ex. 4. Two scale-pans, each of mass 1 lb., are connected by a string which passes over a smooth pulley; on the scale-pans are placed masses of 9 lbs. and 5 lbs. respectively; the system now commences to move: find the acceleration and the pressure of each weight on the scale-pan during the motion.

Here, as before, $10g - T = 10f$
and $T - 6g = 6f$

$$\therefore \quad 4g = 16f; \qquad \therefore \; f = 8 \text{ f/s/s}.$$

Now, each scale-pan may be regarded as a lift moving vertically with an acceleration of 8*f*/*s*/*s*.

In the case of the 9 lbs. mass, we have, if R denote its pressure on the pan :—

$$mg - R = mf, \quad \text{or} \quad 9 \times 32 - R = 9 \times 8;$$

$$\therefore \quad R = 216 \text{ poundals} = 6\tfrac{3}{4} \text{ lbs. wt.}$$

Also, if R' denotes the pressure of the 5 lbs. mass,

$$R' - 5 \times 32 = 5 \times 8;$$

$$\therefore \quad R' = 200 \text{ poundals} = 6\tfrac{1}{4} \text{ lbs. wt.}$$

208. *A mass m hanging freely draws a mass m′ along a smooth horizontal table, the string passing over a pulley at the edge of the table : to find the acceleration and the tension of the string.*

Here, considering the motion of m, we have, as before,

$$mg - T = mf \qquad (1)$$

Now, in the case of m', its weight $m'g$ is balanced by the reaction R of the table, and therefore the only force producing motion is T.

$$\therefore \quad T = m'f. \qquad (2)$$

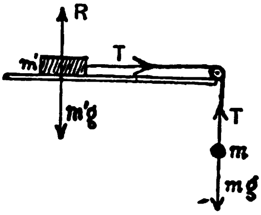


FIG. 97.

Hence, adding (1) and (2), we have

$$mg = (m + m')f; \quad \therefore f = \frac{m}{m + m'} g.$$

Also, substituting this value in (2), we obtain

$$T = \frac{mm'}{m + m'} g \text{ poundals.}$$

Note.—This value of T is exactly half what the tension would be if both were hanging freely in the air (Art. 207).

Ex. A mass of 14 lbs. on a smooth table is connected by a string which, passing over a smooth pulley at the edge of the table, is attached to a mass of 2 lbs. which hangs freely: find the space moved through in the 3rd second after starting from rest; also, find the tension.

Here
$$\begin{aligned} 2g - T &= 2f \\ T &= 14f \\ \hline \text{add} \;\therefore\quad 2g &= 16f; \qquad \therefore\; f = 4f/s/s; \end{aligned}$$

but space in 3rd second = space in 3 seconds − space in 2 seconds

$$= \tfrac{1}{2} \times 4 \times 3^2 - \tfrac{1}{2} \times 4 \times 2^2 = 10 \text{ feet}.$$

Also, $T = 14f = 14 \times 4 = 56$ poundals $= 1\tfrac{3}{4}$ lbs. wt.

209. *If the mass m' rest on a rough horizontal plane whose coefficient of friction is μ, the other mass m hanging freely over the edge, to find the acceleration and tension of the string.*

Here the equation of motion for m is the same as before; but in the case of m', besides the force T which drags it along the table, there is the retarding force of friction which $= \mu \times$ normal pressure $= \mu m'g$; we have, therefore,

$$\begin{aligned} mg - T &= mf \\ \text{and} \qquad T - \mu m'g &= m'f \\ \hline \text{add} \qquad (m - \mu m')g &= (m + m')f; \end{aligned}$$

$$\therefore\; f = \frac{m - \mu m'}{m + m'}\, g;$$

and, on substituting this value of f in either of the above equations, we obtain

$$T = \frac{mm'(1 + \mu)}{m + m'}\, g \text{ poundals}.$$

Ex. A mass of 7 lbs. is on a table and is connected by a string to a mass of 1 lb. which hangs over the edge. If the coefficient of friction be $\frac{1}{14}$, find the time taken to get up a velocity of 15 miles per hour.

Here
$$\begin{aligned} g - T &= f \\ T - \tfrac{1}{14}7g &= 7f \\ \hline \text{add} \qquad g - \tfrac{1}{2}g &= 8f; \qquad \therefore\; f = 2f/s/s. \end{aligned}$$

Now, 15 miles per hour $= 22f/s$;

and $v = ft$ or $22 = 2t$; $\therefore\; t = 11$ seconds.

210. *A mass m hanging freely is attached by a string to a mass m′ on a smooth inclined plane, the string passing over a smooth pulley at the vertex: to find the acceleration and the tension of the string.*

Here the equation of motion for the mass m is clearly the same as before, viz.:—

$$mg - T = mf. \tag{1}$$

Now, the weight of the mass m', viz. $m'g$, resolves into two forces—one perpendicular to the plane, and equal to $m'g\cos\alpha$, and the other along the plane, and equal to $m'g\sin\alpha$. The component perpendicular to the plane $m'g\cos\alpha$ is balanced by the normal reaction R of the plane; therefore, the only forces causing motion are the tension T upwards, and $m'g\sin\alpha$ downwards, which, supposing T is $> m'g\sin\alpha$, gives a resultant $T - m'g\sin\alpha$;

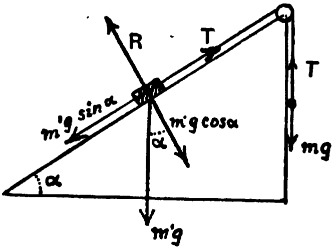


FIG. 98.

$$\therefore\quad T - m'g\sin\alpha = m'f. \tag{2}$$

On adding these two equations we obtain

$$(m - m'\sin\alpha)g = (m + m')f;\quad \therefore\quad f = \frac{m - m'\sin\alpha}{m + m'}g;$$

and, substituting this value of f in equation (2), we have

$$\begin{aligned} T &= m'g\sin\alpha + \frac{m'(m - m'\sin\alpha)}{m + m'}g \\ &= m'g\left(\sin\alpha + \frac{m - m'\sin\alpha}{m + m'}\right), \\ &= \frac{mm'(1 + \sin\alpha)}{m + m'}g \text{ poundals.} \end{aligned}$$

Ex. A mass of 18 lbs. is placed on a smooth inclined plane of inclination 30°, and is connected by a string with a mass of 14 lbs. which hangs freely, the string passing over a smooth pulley at the vertex of the plane; if the 18-lb. mass be projected down the plane with a velocity of 20 feet per second, find how far it will move before its motion is reversed. Also find the tension of the string.

Here the weight of the 18-lb. mass, viz. $18g$, resolves into $18g \cos 30°$ perpendicular to the plane, and $18g \sin 30°$ along the plane; the former component is balanced by the normal reaction of the plane; hence there remains $18g \sin 30°$, i.e. $18g \times \frac{1}{2}$ or $9g$; and as this force is less than the weight $14g$ of the mass which hangs freely, therefore the acceleration will act as a retardation. The two equations are, therefore,

$$14g - T = 14f \qquad (1)$$

$$T - 9g = 18f \qquad (2)$$

$$\therefore\ 5g = 32f; \quad \therefore\ f = 5 f/s/s.$$

Also,

$$v^2 = u^2 + 2fs;$$

$$\therefore\ 0 = 20^2 + 2(-5)s = 400 - 10s;$$

$$\therefore\ s = 40 \text{ feet.}$$

Also, substituting the value of f in equation (2), we have

$$T - 9 \times 32 = 18 \times 5;$$

$$\therefore\ T = 288 + 90 = 378 \text{ poundals} = 11\tfrac{13}{16} \text{ lbs. wt.}$$

211. In the same way it may be shown that if two smooth inclined planes of inclination α and β are placed back to back, and a mass m on the former is connected by a string to a mass m' placed on the latter, then, supposing that m descends and m' ascends, the acceleration and the tension of the string are given by the following equations:—

$$f = \frac{m \sin \alpha - m' \sin \beta}{m + m'} g$$

and

$$T = \frac{mm' (\sin \alpha + \sin \beta)}{m + m'} g \text{ poundals.}$$

212. The student, by following the principles employed in the previous cases, will have little difficulty in deducing these results, and also in applying the same principles to deduce the acceleration and tension of the string when m' is placed on a rough inclined plane and m hangs freely, or when each is placed on a rough inclined plane.

Atwood's Machine.

213. Atwood's machine was originally invented in order to verify, by actual experiment, Newton's Laws of Motion applied to falling bodies. It can also be used to find the value of g at any place, although the results are not at all as accurate as that given by means of the pendulum explained in Chapter XVIII.

214. It consists of a small pulley whose axis C, as it rotates, rolls on the circumferences of several wheels (known as friction wheels), thus reducing the friction to a very small amount. Two bodies, A and B, of equal mass m, are connected by a fine string which passes over the pulley C. For the purpose of measuring the vertical distances moved through there is a vertical graduated scale XY, and for measuring time there is a clock or stop-watch. R is a ring through which the mass A can pass, and which can be clamped in any position. A small bar K called a "rider" is placed on A, so that the total mass of A and K is *slightly* greater than that of B, in order that the resultant motion may be very slow and can therefore be accurately measured. The ring R, being large enough to let the mass A through, stops the rider K, which is therefore lifted off as A goes through, so that A, after passing through the ring, may move with uniform velocity (there being then no impressed force). D is a platform on which A is supported, and which can be instantaneously released when motion is about to take place, and E is a fixed platform at a measured distance below R, which will stop the motion of A.

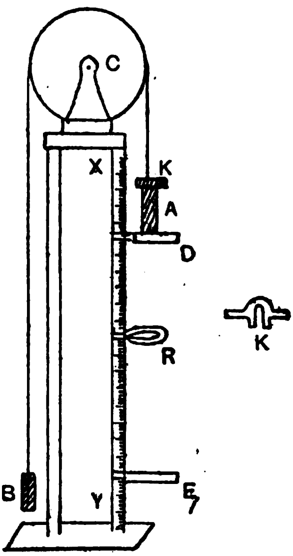


FIG. 99.

215. *To find the value of g at any place by Atwood's machine.*

Let the distance DR through which, when falling, the motion is accelerated be called h; and let the distance RE along which the velocity is uniform be h'. Also let the mass of the rider be denoted by x.

Now, when falling through the height h, the acceleration is given (Art. 207) by the equation

$$f = \frac{(m+x) - m}{(m+x) + m} g = \frac{x}{2m + x} g.$$

Then, if v be the velocity at the moment of passing through the ring, we have:—

$$v^2 = 2fh = \frac{2x}{2m + x} gh.$$

Now, after its passage through the ring, the mass A moves through the distance RE or h' with uniform velocity v; and the time taken to fall through this distance is carefully observed: let this time be denoted by t. But for uniform motion (Art. 160):—

$$\text{space} = \text{velocity} \times \text{time}, \quad \text{or} \quad h' = vt; \quad \therefore \quad h'^2 = v^2t^2, \quad \text{or}$$

$$h'^2 = \frac{2x}{2m + x} ght^2, \quad \therefore \quad g = \frac{h'^2(2m + x)}{2xht^2};$$

and, as all the quantities in the right-hand side of the equation are known, g is therefore found.

216. To prove by means of Atwood's machine that *the acceleration produced is proportional to the impressed force, provided the mass acted on by the force is the same.*

Let the ring R be fixed at a measured distance below the platform on which A with the rider K rests; then the velocity acquired at the moment the rider is lifted off by the ring R is found by noting the time taken to fall with *uniform velocity* the distance RE from the ring to the lower platform E; this distance divided by the time gives the velocity at R, from which, knowing the distance DR, the acceleration during the fall from *D* to *R* can be found from the equation $v^2 = 2fs$ (or indeed from the equation $v = ft$ if the time be noted). In this case the total mass moved was $2m + x$, and the moving force was x. If now another experiment be made, taking such masses that the total mass moved shall still be the same, and the moving force twice what it was before, it will be found that twice the previous acceleration will be produced.

217. We can make this more clear, perhaps, if we illustrate it by numbers. For example, if, in the first instance, *A* and *B* are each masses of 7 ozs., and the mass of the rider *K* be 2 ozs., then the moving force is a weight of 2 ozs., and the total mass moved is 16 ozs., and we obtain the acceleration as above. If now the masses of *A* and *B* are each taken as 6 ozs. and *K* as 4 ozs., the moving force is now double what it was before, viz. a weight of 4 ozs., and the total mass moved is still 16 ozs.; and it is found that the acceleration is double what it was before. Similarly, by repeatedly varying the magnitude of the moving force, we find that the acceleration produced in the same mass is proportional to this force.

218. In the same way it may be proved that *the force is proportional to the mass, provided the acceleration is the same.*

For, if the total mass be doubled, and the moving force be also doubled, it is found that the acceleration is unaltered; thus if, instead of the masses *A* and *B* being each 7 ozs., and the rider *K* 2 ozs., giving a total mass of 16 ozs., we make *A* and *B* each 14 ozs., and *K* 4 ozs., we find that the acceleration is the same as before.

Similarly, if we treble the mass moved and treble the moving force, we still obtain the same result; ∴ &c.

Note.—We shall conclude this chapter with some examples illustrative of Newton's Laws of Motion which are somewhat more difficult than any previously given. We would therefore advise the student who is approaching the subject for the first time not to read them, as they can be conveniently omitted without a breach in the continuity of his work.

Ex. 1. A slider A of mass 39 ozs. is tied by a string to a slider B of mass 26 ozs.; A is sent up a rough plane dragging B after it with a velocity of $56 f/s$: how far up will it slide, taking the inclination of the plane to be $\sin^{-1} \frac{5}{13}$, and the coefficient of friction between A and B and the plane to be respectively $\frac{1}{3}$ and $\frac{1}{2}$? Find also the tension of the string during the motion, and state what will happen when A ceases to move up the plane.

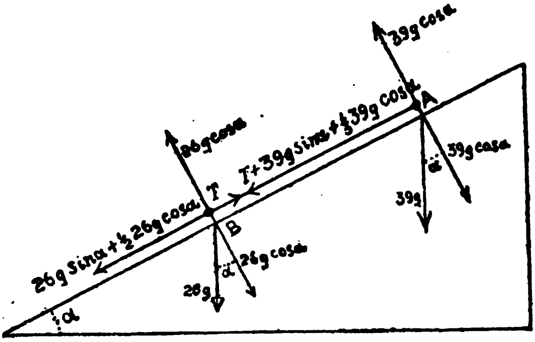


FIG. 100.

Here, taking the unit of mass as the ounce instead of the pound, the weight $26g$ absolute units resolves into $26g \sin \alpha$ along the plane, and $26g \cos \alpha$ perpendicular to the plane.

Similarly, $39g$ resolves into $39g \sin \alpha$ and $39g \cos \alpha$ along and perpendicular to the plane respectively.

The friction acting on the 26-oz. mass is $\frac{1}{2}\, 26g \cos \alpha$, and the friction acting on the 39-oz. mass is $\frac{1}{3}\, 39g \cos \alpha$.

Also, let T be the tension of the string, acting upwards on the 26-oz. and downwards on the 39-oz. mass.

The equations of motion now are

$$26g \sin \alpha + \tfrac{1}{2}\, 26g \cos \alpha - T = 26f,$$

and

$$T + 39g \sin \alpha + \tfrac{1}{3}\, 39g \cos \alpha = 39f;$$

these equations reduce to

$$10g + 12g - T = 26f$$

and

$$\underline{T + 15g + 12g = 39f}$$

add

$$49g = 65f; \quad \therefore \quad f = \frac{49 \times 32}{65}.$$

On substituting the value of f in one of the above equations we find that

$$T = 2\tfrac{2}{5} \text{ ozs. wt.};$$

also

$$v^2 = u^2 + 2fs,$$

$$O = 56^2 - 2\,\frac{49 \times 32}{65}\, s; \quad \therefore \quad s = 65 \text{ feet.}$$

Also, after reaching the highest point, it is evident that the 39-oz. mass will commence to slide down as $39g \sin \alpha$, which $= 15g$, which pulls it down the plane, is greater than $\frac{1}{3}\, 39g \cos \alpha$, which $= 12g$, which retards it. On the other hand, the 26-oz. mass will remain at rest; for $26g \sin \alpha$ or $10g$, which tends to drag it down the plane, is less than $\frac{1}{2}\, 26g \cos \alpha$ or $12g$, which would tend to retard it were motion about to take place.

Ex. 2. In a system of three light pulleys, in which each string is tied to the weight, find the motion when the power is a mass of 2 lbs., and the weight one of 30 lbs., and find the tensions of the strings.

Here, if T be the tension of the string to which the 2-lb. mass is attached, the pulleys being weightless, the tensions of the other strings must be $2T$ and $4T$ respectively. Hence, there is a total force of $7T$ acting vertically upwards on the 30-lb. mass, whose equation of motion is therefore, f being the acceleration with which it is descending,

$$30g - 7T = \cancel{30f}. \qquad (1)$$

Also, if the 30-lb. mass descends with an acceleration of f, the 2-lb. mass must ascend with an acceleration of $7f$; therefore, we have

$$T - 2g = 2 \times 7f; \qquad (2)$$

$\therefore$ from (1) $30g - 7T = 30f$

and from (2) $7T - 14g = 98f$

$$16g = 128f; \quad \therefore \quad f = \frac{g}{8} = 4f/s/s.$$

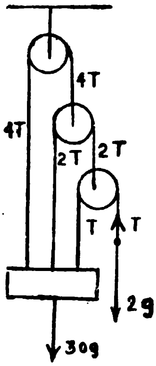


FIG. 101.

Also from (2) $T - 64 = 56$; $\therefore$ $T = 120$ poundals = 3¾ lbs. wt.

Hence the 30-lb. mass descends with an acceleration of $4f/s/s$, and the 2-lb. mass ascends with an acceleration of $28f/s/s$.

Also, the tensions of the strings are 3¾ lbs., 7½ lbs., and 15 lbs. wt. respectively.

EXAMPLES XXIX.

1. Two masses of 7½ and 8½ ozs. are connected by a string passing over a smooth pulley: find the acceleration, the tension of the string, and the space described in the third second from rest.

2. If the heavier weight be 12 ozs., and moves the lighter weight 36 feet in 3 seconds, the two masses being connected by a string passing over a smooth pulley, find the lighter weight and the tension of the string.

3. Two masses are connected by a string passing over a smooth pulley; if in half a second each mass moves through 2 feet, find the ratio of the masses.

4. Two masses of 18 and 14 ozs. respectively are connected by a string which passes over a smooth peg. What is the pressure on the peg during the motion? Also, if each mass were 14 ozs., what would then be the pressure on the peg?

5. Two masses of 8½ and 7½ ozs. are hung over a smooth pulley. Motion now continues for 10 seconds, when the string is cut; find through what distance and for what time the lighter will continue to ascend before it commences to descend.

6. Two masses of 9 and 7 lbs. are connected by a string which passes over a smooth pulley; at the end of 3 seconds, 8 lbs. fall off the 9-lb. mass: find through what distance the 7-lb. mass will now continue to ascend before its motion is reversed.

7. Two strings pass over a smooth pulley; at the extremities of the strings on one side are hung masses of 6 and 4 lbs. respectively, and on the other side both strings are attached to a mass of 6 lbs.: find the distance moved over in the 4th second from rest, and the tensions of the strings.

8. A mass of 10 lbs. on a smooth table is connected by a string to a mass of 6 lbs. which hangs freely over the edge of the table: find (1) the acceleration of the system, (2) the tension of the string, (3) the space described in the third second after starting from rest.

9. If in the last question the 10-lb. mass be projected backwards (i.e. away from the edge of the table) with a velocity of 6 feet per second, this velocity being communicated by means of the string to the 6-lb. mass, how far will motion continue before it is reversed?

10. A mass of 12 lbs. on a smooth table is connected by means of a string to a scale-pan whose mass is 1 lb., and which carries a mass of 3 lbs. placed on it: find the space moved over in 1 second, the tension of the string, and the pressure of the 3-lb. mass on the scale-pan while the motion takes place.

11. A mass of 28 lbs. on a smooth table is connected by two strings which hang over the edge of the table to two masses of 3 lbs. and 1 lb. respectively: find the acceleration with which the system will move, and the tensions of the strings.

12. A mass of 7 ozs. placed on a table is connected with a mass of 1 oz. which hangs freely over the edge; if the 7-oz. mass be at a distance of 18 feet from the edge, find how long it takes to reach the edge, and what its velocity will be when it reaches the edge.

13. A mass of 18 lbs. is on a rough table, and is connected by a string which, passing over a smooth pulley at the edge of the table, carries a mass of 6 lbs. which hangs freely; if the coefficient of friction be $\frac{1}{3}$, find the velocity got up at the end of 1 second, and the tension of the string during motion.

14. A mass of 14 lbs. resting on a rough horizontal table is connected to a string which, passing over a smooth pulley, carries a mass of 2 lbs. which hangs freely; if the system move over 1 foot in 1 second, find the coefficient of friction, and the tension of the string during motion.

15. Two masses each equal to *m* are connected together by a string, one mass being placed on a smooth table, and the other hanging over the edge: find the acceleration of the system. Also, what mass must be taken from one and added to the other in order that the acceleration may become 8 feet per sec. per sec.?

16. A slider weighing 25 grammes, on a smooth plane, whose inclination is $\sin^{-1}\frac{7}{25}$, is tied by a string passing over a smooth pulley at the top of the plane to a mass of 39 grammes hanging freely: find the acceleration, the tension of the string, and the pressure of the 25-gramme mass on the plane during the motion.

17. Two smooth inclined planes whose inclinations are $\sin^{-1}\frac{12}{13}$ and $\sin^{-1}\frac{3}{5}$ respectively are placed back to back; and masses of 13 and 19 lbs. are placed on them respectively: find the velocity acquired in 5 seconds and the tension of the string.

Note.—As the remaining examples are somewhat more difficult than those previously given, we would advise the student who is approaching the subject for the first time to omit them.

18. Two rough inclined planes whose inclinations are $\sin^{-1}\frac{12}{13}$ and $\sin^{-1}\frac{3}{5}$ respectively are placed back to back; and masses of 13 and 19 lbs. respectively are placed on them, and connected by a string passing over a smooth pulley at the common vertex of the planes. Taking the coefficient of friction to be $\frac{1}{3}$ and $\frac{1}{4}$ respectively, find how far the system will move if the mass of 13 lbs. is projected down its plane with a velocity of 7 *f/s*.

19. Arrange four light pulleys so that for equilibrium the power shall be to the weight as 1 to 6. If then the power is a mass of 1 lb., and the weight a mass of 12 lbs., find the velocity acquired by the weight from rest in 2 seconds.

20. In a wheel and axle the radius of the wheel is 3 feet and that of the axle 6 inches, the weight is a mass of 72 lbs., and the power a mass of 14 lbs.: find, neglecting the mass of the wheel and axle, how far the 14 lbs. will descend from rest in 3 seconds.

21. A slider of mass 15 lbs. is sent up a rough inclined plane, whose inclination is $\sin^{-1}\frac{3}{5}$, with a velocity of 29 *f/s*, dragging after it a slider of mass 17 lbs. to which it is tied by a string: how far up will they go, taking the coefficients of friction for the 15 lbs. and 17 lbs. to be $\frac{1}{4}$ and $\frac{1}{2}$ respectively, and what will happen next?

22. A mass *A* of 13 lbs. is tied by a string to a mass *B* of 13 lbs., which is again tied to a mass *C* of 26 lbs. The strings being fully stretched, *A* is sent up a rough plane whose inclination is $\sin^{-1}\frac{5}{13}$ with an initial velocity of 12 *f/s*. Find how far the system will move up the plane, and the tensions of the strings during the motion, taking the coefficients of friction for *A*, *B*, and *C* to be $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$ respectively.

23. Arrange five light pulleys so that for equilibrium the power shall be to the weight as 1 to 16. If, then, the weight is a mass of 33 lbs. and the power a mass of 1 lb., find how far the weight will descend from rest in $2\frac{1}{3}$ seconds.

24. A monkey hangs at the end of a long rope which, passing over a smooth pulley, carries at its other extremity a mass equal to that of the monkey. The monkey now proceeds to climb up the rope: describe what happens to the other mass from the time he commences his ascent until he reaches the pulley.

25. In a system of three light pulleys, in which each string is tied to the weight, the power being a mass of 8 ozs., and the weight a mass of $7\frac{1}{2}$ lbs., find the distance through which the weight descends in 3 secouds, starting from rest: also find the tensions of the strings.

CHAPTER XV.

COMPOSITION OF VELOCITIES AND ACCELERATIONS.

219. **Uniform Velocities represented by Straight Lines.**—In order to completely specify the velocity of a particle, we should state (1) its magnitude, (2) the direction of the motion. If, therefore, in the case of uniform rectilinear motion, we lay down a straight line containing as many units of length as there are feet per second in the velocity (or centimetres per second, as the case may be), and whose direction is the same as that of the motion of the particle, then this line will represent the velocity in magnitude and line of action, an arrow-head representing the direction. The order in which the letters describing it are used will also, as in the case of forces, represent the direction; thus, when a velocity is represented by a straight line AB, the direction of motion is from A to B; whereas, if represented by a line BA, the motion would be from B to A.

220. **Variable Velocities.**—In the case of rectilinear motion, when the particle does not move over equal spaces in equal times, its velocity is said to be variable. In that case, the velocity at any instant may be represented by the straight line indicating the distance which the body would traverse in a unit of time, supposing during that interval of time the velocity had been uniform, and the same as that which it possessed at that instant. Also, when the direction of motion is not a straight line, as, for example, when the particle moves along a curve, even if it moves over equal spaces in equal times, the velocity is still said to be variable, for, although its magnitude does not alter, its direction is changing at each instant.

221. In the case of motion along a curve, the velocity of the particle at any instant may also be represented by a straight line whose direction is a tangent to the curve at the point occupied by the particle at that instant.

222. **Composition of Velocities.**—A person walking along the deck of a steamer, or moving from one position to another of a railway-carriage which is in motion, is an example of a body having what is called two simultaneous velocities. In such cases, the actual velocity of the body is called the *Resultant* velocity, the two simultaneous velocities being called the *Components.*

223. **Parallelogram of Velocities.**—*If a particle has two simultaneous velocities represented in magnitude and direction by two sides of a parallelogram drawn from a point, the resultant velocity is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.*

For let v_1 and v_2 be the two simultaneous velocities represented by the lines OA and OB respectively. Now, let the particle move along the line OA with velocity v_1, and let the line OA at the same time move parallel to itself, so that its extremities O and A move along OB and AC respectively, each with a velocity v_2. At the end of one second, the particle will have moved a distance OA along the direction OA; but the line OA will then of itself have moved into the position BC, so that the particle will actually at the end of the second be at C.

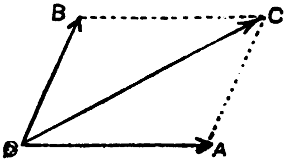


FIG. 102.

But, since the component velocites, being uniform, are at each instant the same in magnitude and direction, the resultant velocity must also be the same at each instant in magnitude and direction; and therefore the diagonal OC represents the resultant velocity.

From this it follows that velocities can be compounded and resolved in the same way as forces; and hence the following propositions are true, the proofs being similar to those given in Arts. 22, 24, and 31:—

(1) *If v be the resultant of two component velocities v_1 and v_2 whose directions are at right angles to one another, then*

$$v^2 = v_1^2 + v_2^2.$$

(2) *If v be the resultant of two component velocities v_1 and v_2 acting at an angle α, then*

$$v^2 = v_1^2 + v_2^2 + 2v_1v_2 \cos \alpha.$$

(3) *Also, if the direction of a velocity v makes an angle θ with a given direction, then v may be resolved into two components, viz.:—*

a component along the given direction which $= v \cos \theta$, and
a component perpendicular to the given direction which $= v \sin \theta$.

Ex. 1. A ship is sailing at the rate of 25 feet per second when a ball is rolled along the deck with a velocity of 15 feet per second in a direction inclined at an angle of 60° with the direction of the ship's course. Determine the velocity of the ball relative to the bottom of the sea, supposed to be horizontal.

Here the ball may be said to have two simultaneous velocities of 25 and 15 feet per second whose directions are inclined at an angle of 60°.

Now, the velocity of 15 feet per second may be resolved into two components, viz.:—

A component along the direction of the ship's motion

$$= 15 \cos 60 = \frac{15}{2},$$

and a component perpendicular to the direction of the ship's motion

$$= 15 \sin 60 = \frac{15\sqrt{3}}{2}.$$

Hence, the ball may be regarded as having two velocities at right angles to one another, viz.:—

$$25 + \frac{15}{2} = \frac{65}{2} \text{ along the direction of the ship's motion,}$$

and

$$\frac{15\sqrt{3}}{2} \text{ perpendicular to the direction of the ship's motion;}$$

$$\therefore\ v = \sqrt{\left(\frac{65}{2}\right)^2 + \left(\frac{15\sqrt{3}}{2}\right)^2} = 35 f/s.$$

Ex. 2. A ferry-boat can steam at the rate of 10 miles per hour on still water: find in what direction her head must be pointed in order that she may cross a river in a direction at right angles to the current, which flows at the rate of 5 miles per hour.

Lay down OA to represent the velocity of the current, viz. 5 miles per hour.

Erect OK perpendicular to OA. With A as centre, and 10 as radius, describe a circle cutting OK in C. Join AC and complete the parallelogram AD. Then OD will be the required direction of the boat's head.

For the boat may be regarded as having two velocities of 10 and 5 miles per hour represented by OD and OA respectively, which give a resultant velocity represented by OC which is perpendicular to the direction of the current. Also we have

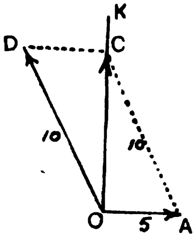


FIG. 103.

$$\cos OAC = \frac{5}{10} = \frac{1}{2};$$

therefore the angle $OAC = 60°$, and the angle $AOD = 120°$.

Hence, the direction of the boat's head must be at an angle of 120° with the direction of the current.

224. **Change of Velocity.**—If a particle be moving at any instant with a velocity represented by OA, and at some subsequent instant its velocity is represented by OC, then *the change in the velocity of the particle during the interval is represented by the third side of the triangle AOC, viz. AC.* For:—

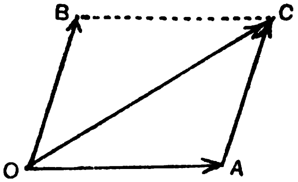


FIG. 104.

Complete the parallelogram $OACB$.

Then velocities represented by OA and OB give a resultant velocity represented by OC; therefore, to change a velocity represented by OA into one represented by OC, an additional velocity represented by OB or AC must be applied. Hence OB or AC represents the change in velocity during the interval.

Also, supposing the interval between the two instants to be t units of time, the average acceleration during the interval is $\frac{AC}{t}$. Hence, we have now a more general definition of acceleration than that previously given for rectilinear motion :—

Acceleration is the rate of change in the velocity, whether the change takes place in the original direction of motion or not.

225. **The Triangle of Velocities.**—*If a particle have simultaneously three velocities which can be represented both in magnitude and direction by the sides of a triangle taken in order, then the particle will remain at rest.*

This follows at once from the last Article.

226. **Polygon of Velocities.**—*If a particle have simultaneously any number of velocities which can be represented in magnitude and direction by the sides of a closed polygon taken in order, the particle will remain at rest.*

The proof of this theorem is the same as that already given (Art. 27) for the Polygon of Forces, velocities being substituted instead of forces.

227. **Graphic Construction of Resultant Velocity.**—As in Art. 29, the resultant of a number of simultaneous velocities can be found *graphically* by drawing lines in succession parallel and equal to the lines representing the component velocities; the side which closes the polygon, reversed in direction, will represent the resultant velocity.

Ex. 1. A cricket-ball is moving due east with a velocity of 30 feet per second when it is struck a blow, in consequence of which its velocity is changed to one of 30 feet per second due north: find the direction and magnitude of the velocity given it by the blow.

Here (fig. 103), let OA represent a velocity of $30 f/s$ due east, and OC that of $30 f/s$ due north, which the ball possesses after the blow; hence OD or AC represents the change in velocity, i.e. the velocity produced by the blow.

Hence, $AC = 30\sqrt{2}$, and the direction of AC is north-west.

Ex. 2. A mass of 10 lbs. is moving due north with a velocity of 20 feet per second; it is acted on by a constant force owing to which, at the end of 5 seconds, it is found to be moving in a direction 60° to the east of north with the same velocity: find (1) the total change in velocity, (2) the acceleration, (3) the magnitude and direction of the force.

Here, on laying down lines to represent velocities of 20*f/s* due north, and 20*f/s* 60° east of north, we see that the third side of the triangle is also 20.

Hence, the change in the velocity during 5 seconds is 20*f/s*, and therefore the acceleration is 4 feet per sec. per sec.; but

$$P = mf \text{ poundals} = 10 \times 4 = 40 \text{ poundals} = 1\tfrac{1}{4} \text{ lbs. wt.}$$

Also, the direction of the force makes an angle of 120° with due north, i.e. 30° south of east.

228. **Definition: Relative Velocity.**—The velocity of one body *relative* to another is that velocity which the first would have to possess while the second body is supposed to be at rest in order that the same change in the relative positions of the bodies may be produced as would be produced by the velocities which the bodies actually possess; or, in other words, it is the velocity which the first body would *appear* to have to an observer moving with the second body.

For example, if two trains travel in opposite directions with velocities of 40 and 30 miles per hour respectively, the relative velocity of each with respect to the other is 70 miles per hour; whereas, if they are moving in the same direction, the relative velocity is 10 miles per hour.

229. *If the velocities of two particles be represented by two sides AB and AC of a triangle, their relative velocity shall be represented by the third side BC.*

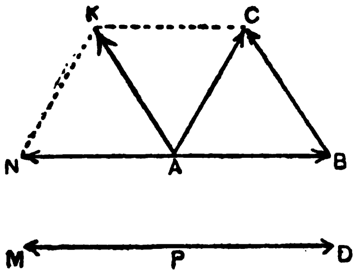


FIG. 105.

For, let one particle at A have a velocity represented by

AC, and the other at *P* with velocity represented by *PD*, which is equal and parallel to *AB*. Apply to both *P* and *A* velocities *PM* and *AN* each equal to *PD*; the particle *P* is now reduced to rest. Complete the parallelogram *AK*; and *AK* will represent the relative velocity of *A* viewed from *P*. But $AK = BC$, the third side of the triangle *ABC*; ∴ &c.

Ex. A man holding an umbrella runs with a velocity equal to the vertical velocity with which rain is falling, in consequence of which the rain strikes him in the face. At what angle should he hold the handle of the umbrella in order to shield himself best?

Lay down *OA* and *OB* to represent the velocities of the man and a raindrop respectively, *OA* being equal to *OB*.

Now, apply to both the man and the raindrop a velocity *OA'* equal and opposite to *OA*. The man will now be brought to rest; and, on completing the parallelogram *OA'CB*, the diagonal *OC* represents the relative velocity of the raindrop. Hence, he must hold the handle of the umbrella in the direction of *OC* produced, viz. *OC'*; and, since the angle $COB = 45°$, therefore he must hold the umbrella before him, so that the handle makes an angle of 45° with the vertical.

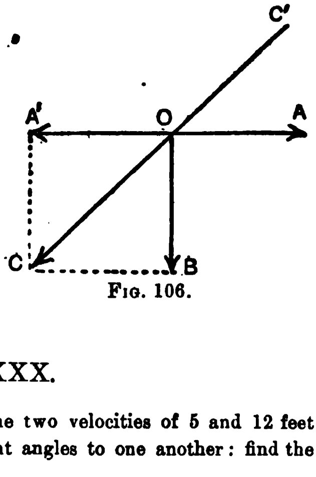


FIG. 106.

EXAMPLES XXX.

1. A particle is given at the same time two velocities of 5 and 12 feet per second respectively in directions at right angles to one another: find the resultant velocity.

2. A ship is sailing at the rate of 15 miles per hour when a ball is thrown along the deck with a velocity equal to that of the ship: find the magnitude of the actual velocity of the ball relative to the bottom of the sea, supposed to be horizontal, (*a*) if the direction in which the ball is thrown is the same as that of the ship's motion, (*b*) opposite to that of the ship's motion, (*c*) if the direction makes an angle of 120° with the direction of the motion of the ship.

3. Rain is falling vertically, but to a cyclist the raindrops strike him in the face, seeming to come at an angle of 30° to the vertical: compare the velocity of the cyclist with that of the drops of rain.

4. A man swimming across a river wishes to go across in the shortest possible distance, and finds that in order to do so he has to swim up the stream in a direction making an angle of 135° with that of the current: compare the speed with which he could swim on still water with that of the current.

5. In what direction should a man swim across a river in order to reach the opposite bank in the shortest possible time?

6. A train is moving with a velocity of 30 miles per hour when a stone is thrown through one of the windows with a velocity equal to that of the train, and in a direction perpendicular to the direction of the motion of the train. What is the apparent direction of the motion of the stone to a passenger whom it strikes, and with what velocity will it strike him?

7. A mass of 20 lbs. is moving with a velocity of 30 feet per second due north when a constant force commences to act on it, and at the end of 5 seconds the body has a velocity of 40 feet per second due east: find the magnitude and direction of the force.

8. A particle of mass 12 lbs. is moving to the north on a smooth horizontal plane with a velocity of 32 feet per second. What force must act on it in order to bring it to a point 72 feet east of its initial position in 3 seconds? Also find the direction of the force.

9. Two cyclists, starting from the same point at the same time, ride along roads at right angles to one another with velocities of 6 and 8 miles per hour respectively: how far will they be apart in half an hour?

10. Two railway trains move along lines at right angles to one another towards a crossing distant one mile from one and half a mile from the other; the former travels at the rate of 40 and the latter of 30 miles per hour. Find their shortest distance apart.

Note.—Apply to each train a velocity equal and opposite to the velocity of the latter; the latter train is now brought to rest. Then find the resultant relative velocity of the other, and the perpendicular distance of the point at which the train is at rest from the line of action of the relative velocity of the other will be their shortest distance apart.

11. A particle of mass 8 lbs., when first observed, has a velocity to the north of 10 *f/s*, and one to the east of 36 *f/s*; after 2 seconds it has a velocity to the north of 50 *f/s*, and one to the west of 60 *f/s*. Assuming the change to be due to the action of a constant force, find its magnitude and direction.

230. **Parallelogram of Accelerations.**—*If a particle move with two simultaneous accelerations represented in magnitude and direction by two sides of a parallelogram drawn from a point, the resultant acceleration is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.*

Let P be the position of the particle at the beginning of an interval of 1 second, and let the accelerations be represented by the sides OA and OB of the parallelogram OC.

Then lay down *PQ* to represent the velocity possessed by the particle at the commencement of the 1-second interval. With *PQ* as diagonal, draw a parallelogram *PSQT*, with the sides *PS* and *PT* parallel to *OA* and *OB* respectively; produce *PS* to *M* and *PT* to *N*, so that *SM* is equal to *OA* and *TN* to *OB*. Complete the parallelogram *PR*.

Then the initial velocity *PQ* is equivalent to the two component initial velocities *PS* and *PT*. But in one unit of time *SM* and *TN* represent the respective changes in these component velocities.

Hence, at the end of one unit of time, the component velocities may be represented by *PM* and *PN*, which are equivalent to a resultant velocity *PR*.

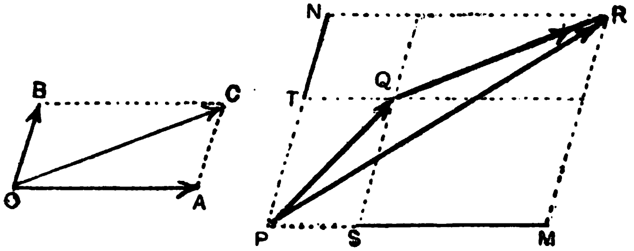


FIG. 107.

But the resultant velocity *PR* is equivalent to the velocities *PQ* and *QR* (Art. 225).

Hence *QR* represents the change in the original velocity *PQ* in one unit of time, i.e. *QR* is the resultant acceleration.

But *QR* is equal and parallel to *OC*. Hence the diagonal *OC* of the parallelogram formed by the lines *OA* and *OB* represents the resultant acceleration in magnitude and direction.

231. From the Parallelogram of Accelerations may be deduced the Triangle of Accelerations and the Polygon of Accelerations, which may be stated as in Arts. 225, 226.

To deduce the Parallelogram of Forces from the Parallelogram of Accelerations.

232. Let two forces P and Q, represented by OA and OB respectively, act on a particle at O; and let f_1 and f_2, represented by OA' and OB', be the respective accelerations produced by these forces. Then, the mass of the particle being m, we have

$$P = mf_1, \quad \text{and} \quad Q = mf_2;$$

$$\therefore \quad \frac{P}{Q} = \frac{mf_1}{mf_2} = \frac{f_1}{f_2};$$

hence,

$$OA : OB :: OA' : OB'.$$

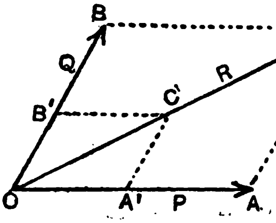


FIG. 108.

Complete the parallelograms OC' and OC; then these two parallelograms have their sides proportional, and the angle AOB common to both; hence (Euclid VI. 26) the points O, C', and C are in the same straight line.

Hence $\qquad OA : OA' :: OB : OB'$

$\qquad\qquad\qquad\qquad :: OC : OC'$;

but OC' represents the resultant of the accelerations f_1 and f_2; therefore, OC represents the resultant of the forces P and Q.

CHAPTER XVI.

UNIFORM MOTION IN A CIRCLE. PROJECTILES.

233. **Angular Velocity.**—If a point P, describing a curve round a point O, move through an arc PQ in t units of time, then, if the angle $POQ = \theta$, we have, supposing the angular motion about O to be uniform,

$\frac{\theta}{t}$ = the angle described about O in each unit of time,

which is called the *angular velocity* about O.

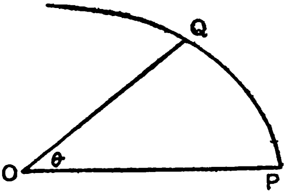


Fig. 109.

Hence, denoting this angular velocity by ω, we have

$$\omega = \frac{\theta}{t}.$$

For example, if a body moving uniformly in a circle describes a complete revolution in 1 minute, its angular velocity is got by the equation

$$\omega = \frac{\theta}{t} = \frac{360^\circ}{60} = 6^\circ \text{ per second};$$

or, expressed in circular measure,

$$\omega = \frac{\theta}{t} = \frac{2\pi}{60} = \frac{\pi}{30} \text{ of a radian per second.}$$

234. **Uniform Motion in a Circle.**—If a body move in a circle with a velocity whose magnitude is constant, then the angular velocity is given by the equation

$$\omega = \frac{v}{r},$$

where v is the velocity of motion, and r the radius of the circle.

For (fig. 109) if PQ be the arc of a circle described in 1 unit of time about the centre O, then the length of PQ represents the magnitude of the velocity v, and the angle $POQ = \omega$.

Hence ω (expressed in circular measure) $= \frac{PQ}{r} = \frac{v}{r}$.

Ex. 1. A body describes a circle of radius 30 feet with a uniform velocity of 10 feet per second: find the angular velocity.

$$\omega = \frac{v}{r} = \frac{10}{30} = \frac{1}{3} \text{ of a radian per second.}$$

Ex. 2. If the angular velocity of a body moving in a circle be 2 radians per second, and the radius of the circle be 30 feet, find the velocity of motion.

$$\omega = \frac{v}{r}; \quad \therefore \quad v = \omega r = 2 \times 30 = 60 \text{ feet per second.}$$

235. **Centripetal Force.**—We know from Newton's First Law that a body tends to move uniformly in a straight line unless acted on by some external impressed force. Now, when a body describing a circle moves over arcs of equal length in equal times, although the magnitude of the velocity is constant, its direction is continually changing, and hence it moves with an acceleration.

If a particle describes a circle of radius r with a velocity of constant magnitude v, the acceleration is towards the centre of the circle, and is given by the equation

$$\mathbf{Of} = \frac{\mathbf{v}^2}{\mathbf{r}}.$$

Let A be the position of the particle at any instant, and B its position at the end of t units of time.

Draw OC and OD parallel to the tangents at A and B respectively; hence, OC is perpendicular to OA, and OD to OB.

Cut off OC and OD to represent the velocities at A and B respectively; hence, $OC = OD = v$.

Then CD represents the change in the velocity in t units of time.

Again, $\angle COD = \angle AOB = \theta$, both having a common complement COB; hence the triangles AOB and COD, having an angle in each equal and the sides containing them proportional, are similar to one another.

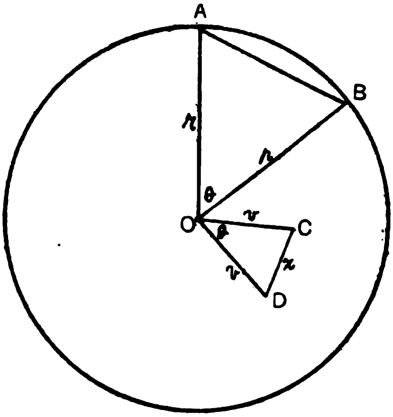


Fig. 110.

But OC and OD are perpendicular to OA and OB respectively; therefore CD is perpendicular to AB. That is, *the change of velocity in any time is always perpendicular to the chord of the arc described during that time.*

Hence, in the limit, when the time is very small, the chord of the very small arc described coincides with the arc itself; and hence the change in the velocity is perpendicular to the arc, and therefore passes through the centre. And, as this is true at all parts of the circumference, it follows that at each instant the acceleration is towards the centre.

Again, if $CD = x$ and $AB = s$, we have

$$x = \text{change of velocity in } t \text{ units of time};$$

$$\therefore \text{ acceleration } f = \frac{x}{t}.$$

But, in the limit, when θ is very small, x may be regarded as the arc of a circle of radius v; therefore, $x = v\theta$;

$$\therefore f = \frac{v\theta}{t} = \frac{v}{t} \cdot \frac{s}{r}; \text{ but } s = vt; \quad \therefore f = \frac{v^2t}{tr} = \frac{v^2}{r}.$$

Corollary.—If ω be the angular velocity about the centre, $v = \omega r$; and substituting this value of v, we obtain $f = \omega^2 r$.

236. Since at each instant the body is subjected to an acceleration towards the centre, it follows from Newton's Second Law that at each instant it is acted on by a force directed towards the centre, the magnitude of this force P being given by the equation

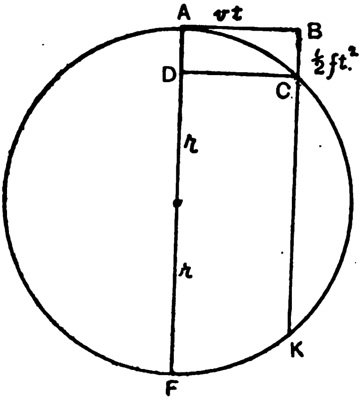


FIG. 111.

$$P = \mathbf{mf} = \frac{\mathbf{mv^2}}{\mathbf{r}} \text{ absolute units.}$$

This force is sometimes called *Centrifugal* force, as if a force acted on the body whose direction was *away from* the centre. It is more properly called *Centripetal force*, i.e. a force acting *towards* the centre.

Another proof of the equation

$$\mathbf{f} = \frac{\mathbf{v}^2}{\mathbf{r}}.$$

237. If, when the particle is at A (fig. 111), there were no force towards the centre, it would fly off along the tangent AB with uniform velocity v. Then, if AB be the space described in t units of time, we have (Art. 161) $AB = vt$.

But, in t units of time, the body is pulled towards the centre by the acceleration f, a distance AD, which $= \frac{1}{2}ft^2$. Complete the parallelogram AC, and C will be the position of the body. Produce BC to K.

But (Euclid III. 36) we have $BK \cdot BC = AB^2$, and, in the limit, when the units of time are very small

$$BK = AF = 2r, \text{ and } BC = AD = \tfrac{1}{2}ft^2; \therefore 2r \times \tfrac{1}{2}ft^2 = v^2t^2,$$

or
$$rf = v^2; \quad \therefore \quad f = \frac{v^2}{r}.$$

238. Examples of centripetal force are matters of everyday experience; for instance, if a person tie a body to the end of a string, and, holding the other end in his hand, cause the body to describe a circle on a horizontal smooth table, then the tension on the string acting towards his hand is the centripetal force which causes the body to describe a circular path. Also, if he lets the string go, the body at once flies off at a tangent to the circle at the part of the circumference which it occupied when set free.

Ex. 1. A mass of 40 lbs. is whirled round in a circle on a horizontal smooth table with a velocity of 12 feet per second, being attached to a fixed point in the table by means of a string 9 feet long: find the tension of the string.

Here the tension

$$P = \frac{mv^2}{r} = \frac{40 \times 12^2}{9} = 640 \text{ poundals} = 20 \text{ lbs. wt.}$$

Ex. 2. A truck of mass 40 tons moves in a curve of 1 mile radius: what is the maximum limit of speed in order that the horizontal reaction exerted by the rails should not exceed $\frac{11}{24}$ ton?

Here $\frac{11}{24} \times 2240 \times 32$ poundals is the centripetal force, and

$$\frac{mv^2}{r} = \frac{40 \times 2240 \times v^2}{5280} \text{ poundals};$$

$$\therefore \quad \frac{11}{24} \times 32 = \frac{40v^2}{5280} \quad \therefore \quad v = 44 f/s = 30 \text{ miles per hour.}$$

Centripetal Acceleration in terms of the Periodic Time.

239. Let T = the Periodic Time, i.e. the time taken to make a complete revolution of the circle.

Now, for uniform motion, $s = ut$;

$$\therefore \quad 2\pi r = vT; \quad \therefore \quad v = \frac{2\pi r}{T},$$

and

$$f = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

and the centripetal force $P = mf = m\,\dfrac{4\pi^2 r}{T^2}$.

Ex. A mass of 1 lb. is whirled round in a circle on a horizontal smooth table at the end of a string 1 yard long, making 4 revolutions per second: find the tension of the string (π^2 may be taken as 10).

Here

$$P = mf = m\,\frac{4\pi^2 r}{T^2} = 1 \,.\, \frac{\overset{120}{4 \times 10 \times 3}}{\left(\frac{1}{4}\right)^2}\ \frac{1}{16} = 1920 \text{ poundals} = 60 \text{ lbs. wt.}$$

Note.—In the following Examples π^2 may be taken as 10.

EXAMPLES XXXI.

1. When a circus rider stands on a horse's back and the horse runs round a circular ring, the rider usually leans towards the centre of the ring. Why does he do so?

2. A mass of 4 lbs. is moving in a circle of 80-feet radius with a velocity of 20 feet per second: find the acceleration towards the centre, and the centripetal force in pounds.

3. A mass of 64 tons is moving in a circle of radius 128 feet with a velocity of 20 feet per second: find the force acting towards the centre which keeps it from flying off at a tangent to the circle.

4. A body whose mass is 5 lbs. describes a circle on a smooth horizontal table round a fixed point to which it is attached by a string 9 feet long with a velocity of 12 feet per second: find the tension of the string.

5. A body whose mass is 300 lbs. is constrained to move in a circle whose radius is 96 feet by means of a string capable of sustaining a tension not greater than 25 lbs.: find the velocity of the body when the string is on the point of breaking.

6. A mass of 10 lbs. moving in a circle 1 yard radius completes 45 revolutions in a minute: find the centripetal force in pounds.

7. A body of mass 1 lb. swings on a horizontal smooth plane round a fixed point to which it is attached by a string 10 feet long. If it make a complete revolution in 10 seconds, find the magnitude and direction of the acceleration and the tension of the string.

8. A mass of 16 lbs. revolves in a circle on a horizontal smooth table at the extremity of a string 1 yard long; if the tension of the string be 15 lbs. wt., find the periodic time (i.e. the time of making 1 revolution).

9. One end of a string 3 yards long is fixed, and the other end is attached to a stone whose mass is 1 lb.; if the stone be whirled round horizontally, prove that, when the tension of the string is 2 lbs. wt., the time of revolution is $\frac{3\pi}{4}$ seconds.

10. The distance of the Moon from the Earth is 240,000 miles, and its periodic time 27 days: calculate the ratio of the force retaining it in its orbit to the force of gravity on the Earth's surface.

11. When a can containing water is quickly swung round in a vertical circle, it is found that the water does not fall out. Explain this.

12. A body describes a circle on a horizontal smooth table at the extremity of a string 2 feet long, the other end of which is fixed; if the tension of the string be 9 times the weight of the body, find the velocity.

13. A string passing through a small hole in a smooth horizontal table has a small sphere of mass m attached to each end of it. The upper sphere revolves in a circle on the table, when it suddenly loses half its velocity: find what diminution must be made in the mass of the lower sphere in order that the upper sphere may continue to revolve in a circle.

Note.—Students reading the book for the first time are advised to omit the remaining examples.

14. A particle of mass 4 ozs., tied by a string 4 feet long to a point on a rough horizontal table, is set in motion with a velocity of 40 feet per second perpendicular to the stretched string: find the initial tension of the string, and also how far the particle will move before it comes to rest, taking the coefficient of friction to be $\frac{1}{4}$.

15. A particle of mass 6 ozs. is tied by a string 4 feet long, fully stretched to a point on a smooth horizontal table, and the string can only stand a tension of 48 ozs. wt. If a constant force of 12 ozs. wt. act on the particle in a direction always perpendicular to the string, find how far the particle will move in its circular path from rest before the string breaks.

16. A particle of mass 12 ozs. is tied by a string 2 feet long to a fixed point. If the particle is allowed to descend from rest in a position in which the string is horizontal and fully stretched, find how far it will have descended by the time the string breaks, as it can bear a tension of only 18 ozs. wt.

17. Taking the radius of the Earth to be 4000 miles, prove that the diminution of the acceleration of gravity at the equator (due to the Earth's rotation on its axis) is approximately ·11 ft. per second per second, given that the periodic time of the Earth's rotation on its axis is 23 hours 56 minutes 4 seconds.

18. Prove that the diminution in the value of g (due to the Earth's rotation on its axis) at a place whose latitude is l is

$$\frac{4\pi^2 R\cos^2 l}{T^2},$$

where R is the radius of the Earth, and T the period of the Earth's rotation on its axis.

Projectiles.

240. When a number of forces act simultaneously on a body, we know from the Second Law of Motion that each will produce the same rate of change of momentum *as if it acted alone* (Art. 197), and, hence, by considering the effect of each force separately, as if the others did not act, the total effect can be calculated.

Thus, when a body is projected into the air with a velocity u in a direction making an angle a with the horizon (fig. 112), the initial velocity u can be resolved into:—

a horizontal component which $= u \cos a$,
and a vertical component which $= u \sin a$;

and we may consider the horizontal and vertical motions separately, calculating each as if the other did not exist; and, by combining the results, many problems can be solved without having to consider, or even be aware of, the properties of the *Parabola*, which is the name of the curve described by the body while in the air.

241.—**Definition.**—*The Angle of Elevation* of a projectile is the inclination of the direction of projection to the horizontal plane through the point of projection.

Thus the angle a is the angle of elevation (fig. 112).

Definition.—The *Range* of a projectile, with reference to a given plane through the point of projection, is the distance, measured along the plane, between the point of projection and the point where the projectile strikes this plane.

When the particular plane in question is a horizontal plane, the Range is called the *Horizontal Range.* Thus OR is the Horizontal Range (fig. 112).

Definition.—The *Time of Flight* of a projectile is the interval that elapses between its projection and when it strikes the plane.

In what follows the resistance of the air on the projectile is neglected, so that the results obtained are not what would actually be true in practice.

A Body is projected on a Horizontal Plane, with a velocity u, at an angle of elevation a, to find (1) *the Time of Flight,* (2) *the Horizontal Range,* (3) *the Greatest Height reached.*

242. Resolving the velocity u horizontally and vertically, we have :—

The horizontal component $= u \cos a$.
The vertical component $= u \sin a$.

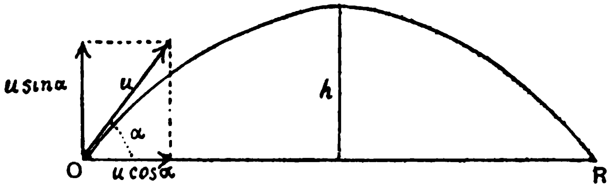


FIG. 112.

Also, the horizontal motion is uniform as the acceleration $= 0$; the vertical motion is accelerated, the acceleration being $= -g$.

243. **The Time of Flight.**—This time is twice the time taken to reach the highest point (Art. 180).

Now, at the highest point the vertical velocity $= 0$; therefore, considering the vertical motion as if the horizontal motion did not exist, we have, making use of the general formula $v = u + ft$,

$$O = u \sin a - gt, \quad \text{or} \quad t = \frac{u \sin a}{g};$$

$$\therefore \text{ whole time of flight } T = \frac{2u \sin a}{g}.$$

244. **The Horizontal Range.**—Consider now the horizontal motion as if the vertical motion did not exist, i.e. we imagine the body to move along the horizontal line OR with a uniform velocity of $u \cos a$ in the time T; making use of the general formula $s = ut$, we have

$$\text{Horizontal Range } OR = u \cos a \, T,$$

$$= u \cos a \times \frac{2u \sin a}{g},$$

$$= \frac{u^2 2 \sin a \cos a}{g} = \frac{u^2 \sin 2a}{g}.$$

245. **Greatest Height.**—At the highest point its vertical velocity = 0, and, making use of the general formula $v^2 = u^2 + 2fs$; $\therefore\ 0 = u^2 \sin^2 a - 2gh$, where h is the greatest height;

$$\therefore \text{ greatest height } h = \frac{u^2 \sin^2 a}{2g}.$$

246. *To find the Angle of Elevation which, with a given velocity of projection, will give the Maximum Horizontal Range.*

$$\text{Generally the range} = \frac{u^2 \sin 2a}{g}.$$

And, u and g being constant, the range will therefore be a maximum when $\sin 2a$ is greatest—i.e. when

$$2a = 90°, \quad \text{or} \quad a = 45°.$$

Hence, an Angle of Elevation of 45° will give the Maximum Range, and

$$\text{The Maximum Range} = \frac{u^2 \sin 90}{g} = \frac{u^2}{g}.$$

Ex. 1. A stone is fired horizontally from the top of a tower 144 feet high with a velocity of 72 feet per second: find (1) the distance from the foot of the tower of the point where it hits the ground; (2) the velocity with which it strikes the ground; (3) the angle at which it strikes the ground.

The initial vertical velocity of the stone = 0.

Hence, considering the vertical motion alone,

$$s = \tfrac{1}{2}ft^2, \quad \text{or} \quad 144 = 16t^2; \quad \therefore \quad t = 3.$$

Also, considering the horizontal motion which is uniform, we have

$$s = ut, \quad \text{or} \quad s = 72 \times 3 = 216 \text{ feet};$$

hence it strikes the ground at a distance of 216 feet from the foot of the tower.

Also, when it strikes the ground, its vertical velocity

$$= 32 \times 3 = 96 \text{ (downwards)} = -96,$$

and its horizontal velocity $= 72$.

Hence the required velocity $= \sqrt{96^2 + 72^2} = 120$ feet per second.

Again, if θ be the angle which the direction of its motion makes with the horizontal, we have

$$\tan \theta = \frac{-96}{72} = -\frac{4}{3}; \quad \therefore \quad \theta = \tan^{-1} - \frac{4}{3}.$$

Ex. 2. A body is projected with a velocity of $200 f/s$ at an angle of elevation of $\sin^{-1}\frac{4}{5}$: find the time of flight, the horizontal range, and the greatest height reached. What is its velocity at the highest point?

Here the horizontal component

$$= 200\cos\alpha = 200 \times \frac{3}{5} = 120 f/s,$$

and the vertical component

$$= 200\sin\alpha = 200 \times \frac{4}{5} = 160 f/s.$$

Now, considering the vertical motion alone, we have, at the highest point,

$$0 = 160 - gt = 160 - 32t;$$

$$\therefore\ t = 5 \text{ seconds};$$

$$\therefore \text{ the whole time of flight } T = 2t = 10 \text{ seconds}.$$

Again, considering the horizontal motion alone, we have

$$s = uT,$$

or the horizontal range $= 120 \times 10 = 1200$ feet.

Also, to find the greatest height, we have

$$v^2 = u^2 + 2fs,$$

$$\text{or } 0 = 160^2 - 64h;\ \therefore\ h = 400 \text{ feet}.$$

Its velocity at the highest point is evidently the horizontal component, viz. $120 f/s$.

Ex. 3. A body is projected at an angle of 30° to the horizon with a velocity of 80 feet per second. Where will it be at the end of 4 seconds?

Here, the horizontal component $= 80\cos 30° = 40\sqrt{3}$,
and the vertical component $= 80\sin 30° = 40$.

Now, consider the vertical motion, using the formula

$$s = ut + \tfrac{1}{2}ft^2,$$

$$\therefore\ s = 40 \times 4 - \tfrac{1}{2} \cdot 32 \cdot 4^2 = -96 \text{ feet}.$$

Hence, at the end of 4 seconds, the body will be at a vertical distance of 96 feet *below* the point of projection.

Also, considering the horizontal motion which is uniform,

$$s = ut = 40\sqrt{3} \times 4 = 160\sqrt{3} \text{ feet}.$$

Hence, at the end of 4 seconds, the horizontal distance traversed will be $160\sqrt{3}$ feet, which, with the vertical distance traversed, fixes the position of the body.

Note.—The remainder of these Examples which are worked out may be omitted by the student on first working over the book.

Ex. 4.—Find the vertical and horizontal components of the velocity with which a ball must be projected from the top of a cliff 140 feet high, so that it shall strike the water in 2 seconds, and in a direction inclined to the perpendicular to the surface at an angle $\tan^{-1}\frac{1}{3}$.

Let u = horizontal component of velocity,
and u' = vertical component of velocity.

Now, considering the vertical motion separately, remembering that s here is -140, we have

$$s = ut + \tfrac{1}{2}ft^2, \quad \text{or} \quad -140 = u' \times 2 - 16 \times 4;$$

$$\therefore \; u' = -38.$$

Also, when the body strikes the water, if θ be the angle which the direction of its motion makes with the perpendicular, we have

$$\tan\theta = \frac{\text{horizontal velocity}}{\text{vertical velocity}}, \quad \text{or} \quad \frac{1}{3} = \frac{u}{\text{vertical velocity}};$$

$$\therefore \; \text{vertical velocity} = 3u, \text{ and is minus in sign.}$$

Hence

$$-3u = u' + ft, \quad \text{or} \quad -3u = -38 - 32 \times 2;$$

$$\therefore \; u = 34.$$

Hence the vertical component is *downwards* and $= 38 f/s$, and the horizontal component is $34 f/s$.

Ex. 5. If a ball can just pass through a ring, find the way in which it must be projected so that it shall pass through the ring when the centre of the ring is distant 315 feet horizontally, and 28 feet vertically upwards from the initial position of the centre of the ball and the plane of the ring at $\tan^{-1}\frac{5}{12}$ to the horizon.

Let u = horizontal component of velocity of projection,
and u' = vertical component of velocity of projection;

$$\therefore \; ut = 315; \tag{1}$$

also,

$$u't - 16t^2 = 28. \tag{2}$$

Also, since the horizontal component of the velocity when the ball passes through the ring is u, the vertical component is $-\frac{12u}{5}$;

$$\therefore \; -\frac{12u}{5} = u' - 32t. \tag{3}$$

Solving for u, u', and t from equations (1), (2), and (3), we obtain

$$t = 7 \text{ seconds}, \quad u = 45 f/s, \quad \text{and} \quad u' = 116 f/s;$$

also, the angle of elevation is

$$\tan^{-1}\frac{u'}{u} = \tan^{-1}\frac{116}{45}.$$

Properties of the Parabola.

247. We shall now consider some of the properties of the curve traced out by a projectile. This curve, called a *parabola*, is one of a group of three curves known as *Conic Sections*, the ellipse and hyperbola being the other two members of the group.

Definitions.—A *parabola* is a curve such that the distance of any point on the circumference from a fixed point, called the *focus*, is equal to its perpendicular distance from a fixed line called the *directrix*.

In other words, a parabola is the locus of a point such that its distance from a given point is equal to its distance from a given line.

Thus, if P be any point on the curve, F the focus, MN the directrix, and PK the perpendicular distance of P from MN, then $PF = PK$.

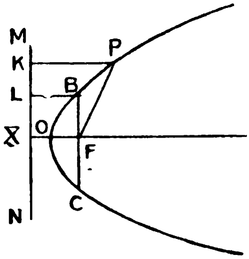


FIG. 113.

Definitions.—The line FX drawn through the focus perpendicular to the directrix is called the *axis*, and the point O where the axis cuts the curve is called the *vertex* of the parabola. The chord BC drawn through F perpendicular to the axis is known as the *latus rectum*, FB or FC being the *semi-latus rectum.*

It is obvious from the definition of the parabola that the semi-latus rectum $BF =$ the perpendicular $BL = FX = 2OX$ or $2OF$ (for $OX = OF$ from the definition of the parabola). Hence the latus rectum $BC = 4OX = 4OF$.

To find the Equation of a Parabola.

248. Let QR be the directrix, and F the focus (fig. 114); then the perpendicular FX drawn from F to the directrix will be the axis of the parabola. Now take as the origin the vertex O, and, as the axes of coordinates, the axis OG of the parabola and the tangent MN at the vertex. Then, if P be any point on the curve, let the lengths PD and PG of the perpendiculars drawn from P to the axes of coordinates be denoted by x and y respectively. Also produce PD to

meet the directrix in K. Then, if the distances OF or OX be each denoted by m, we have

$$PK = PF = x + m, \quad \text{and} \quad FG = x - m.$$

Now $$FG^2 + PG^2 = FP^2 \text{ (EUCLID I. 47)},$$

or $$(x - m)^2 + y^2 = (x + m)^2,$$

or $$x^2 - 2mx + m^2 + y^2 = x^2 + 2mx + m^2,$$

or $$y^2 = 4mx,$$

which is the equation of the parabola, the vertex being the origin.

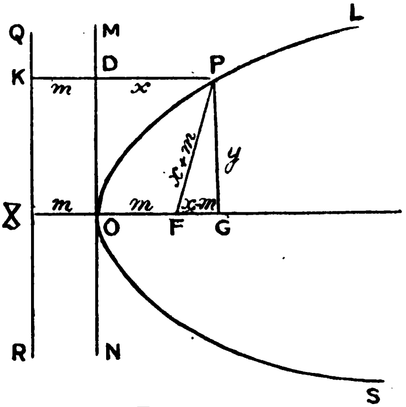


FIG. 114.

The parabola is not a closed curve, as the branches OL and OS can be continued indefinitely without meeting.

From the above equation we have

$$\frac{y^2}{x} = 4m = \text{a constant quantity} = k.$$

Hence we see that if a variable point move in such a way that

$$\frac{y^2}{x} = \text{a constant quantity } k,$$

i.e. so that the square of its distance from one of the axes of coordinates is in a constant ratio to its distance from the other axis, the locus of the point is a parabola whose vertex is at the origin, the axes of coordinates being the axis of the parabola and the tangent at the vertex. Also, since $k = 4m$, we see that the distance m of the focus to the vertex $= \dfrac{k}{4}$.

To show that the Path of a Projectile is a Parabola.

249. Let A be the point of projection of a particle, the velocity of projection being u, whose direction is inclined at an angle α to the horizon, the horizontal and vertical components of u being $u\cos\alpha$ and $u\sin\alpha$ respectively.

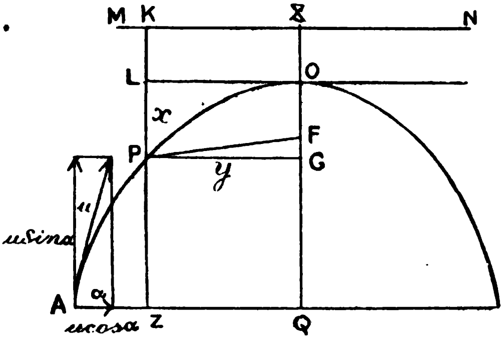


FIG. 115.

Now, let P be the position of the particle at the end of t seconds, O the highest point of the curve, and OL the tangent to the curve at O.

Draw a vertical OQ through O, and drop perpendiculars PL and PG on OL and OQ respectively, PL cutting the horizontal line through A in Z. Let $PL = x$ and $PG = y$; we now have

AZ = horizontal distance described in t secs. $= ut\cos\alpha$;

but

AQ = half the horizontal range $= \dfrac{u^2\sin\alpha\cos\alpha}{g}$; (Art. 244.)

$$\therefore\ ZQ \text{ or } y = AQ - AZ = \frac{u^2\sin\alpha\cos\alpha}{g} - ut\cos\alpha;$$

$$\therefore\ y = \frac{u^2\sin\alpha\cos\alpha - g\,ut\cos\alpha}{g};$$

$$\therefore\ y = \frac{u\cos\alpha}{g}(u\sin\alpha - gt).$$

Also PZ = vertical distance described in t secs.

$$= ut\sin\alpha - \tfrac{1}{2}gt^2,$$

and $$LZ = OQ = \text{greatest height} = \frac{u^2\sin^2\alpha}{2g};$$

$$\therefore\quad PL \text{ or } x = LZ - PZ = \frac{u^2\sin^2\alpha}{2g} - ut\sin\alpha + \tfrac{1}{2}gt^2$$

$$= \frac{u^2\sin^2\alpha - 2ugt\sin\alpha + g^2t^2}{2g}$$

$$= \frac{(u\sin\alpha - gt)^2}{2g}.$$

Hence $$\frac{y^2}{x} = \frac{u^2\cos^2\alpha}{g^2}(u\sin\alpha - gt)^2 \div \frac{(u\sin\alpha - gt)^2}{2g}$$

$$= \frac{2u^2\cos^2\alpha}{g};$$

but $\dfrac{2u^2\cos^2\alpha}{g}$, being independent of t, is a constant quantity, having the same value for all positions of P. Let this constant $= K$;

$$\therefore\quad \frac{y^2}{x} = K.$$

Hence the locus of P is a parabola, with its vertex at O, the axis being OQ.

The focus F is found by measuring off

$$OF = \tfrac{1}{4}K = \frac{u^2\cos^2\alpha}{2g}.$$

To find the height of the Directrix above the Point of Projection.

250. Produce FO to X (fig. 115) so that

$$OX = OF = \frac{u^2\cos^2\alpha}{2g};$$

then MN drawn perpendicular to OX at X is the directrix. Produce PL to K.

Hence the height of the directrix above A is KZ; but

$$KZ = XQ = OQ + OX$$

$$= \frac{u^2 \sin^2 a}{2g} + \frac{u^2 \cos^2 a}{2g}$$

$$= \frac{u^2}{2g}(\sin^2 a + \cos^2 a)\text{; but } \sin^2 a + \cos^2 a = 1;$$

$$\therefore \quad \text{required height} = \frac{u^2}{2g}.$$

But if the body were projected *vertically* with a velocity u, the height h to which it would rise would be $\frac{u^2}{2g}$ (for $0 = u^2 - 2gh$). Hence it follows that the height of the directrix above the point of projection is the height to which the body would have risen if projected vertically with the original velocity u.

Corollary.—From this it follows that if a number of bodies be projected from the same point in different directions, but with the same velocity, the parabolas described by them would all have the same directrix, for in each case the height of the directrix above the point of projection is the same, viz. $\frac{u^2}{2g}$.

To find the Velocity of a Projectile at any Point.

251. Let v be the velocity at any point P after t seconds. Now v is the resultant of—

a horizontal component which $= u \cos a$,

and a vertical component which $= u \sin a - gt$.

$$\text{Hence} \quad v^2 = (u \cos a)^2 + (u \sin a - gt)^2$$

$$= u^2 \cos^2 a + u^2 \sin^2 a - 2ugt \sin a + g^2t^2$$

$$= u^2(\sin^2 a + \cos^2 a) - 2ugt \sin a + g^2t^2;$$

$$\therefore \quad v^2 = u^2 - 2ugt \sin a + g^2t^2;$$

$$\therefore \quad v = \sqrt{u^2 - 2ugt \sin a + g^2t^2}.$$

It can be easily shown that this velocity is the same as that which it would have acquired by a fall through a height equal to that of the directrix above the point P (fig. 115).

For if v be the velocity acquired by a fall through the height KP, then

$$\begin{aligned} v^2 &= 2gKP \text{ (fig. 115)} \\ &= 2g\,(KZ - PZ) \\ &= 2g\left\{\frac{u^2}{2g} - (ut\sin\alpha - \tfrac{1}{2}gt^2)\right\} \\ &= u^2 - 2ugt\sin\alpha + g^2t^2; \\ \therefore\quad v &= \sqrt{u^2 - 2ugt\sin\alpha + g^2t^2}: \end{aligned}$$

the same result as before.

To find the Range of a Projectile on an Inclined Plane, the path of the projectile being in the same plane as the line of greatest slope of the inclined plane.

252. Let the body be projected from the point A as before with velocity u, whose direction makes an angle α with the horizon AC. Also let the inclination of the plane AB to the horizon be β. Then, denoting the range AB by x, and the time of flight by t, we have

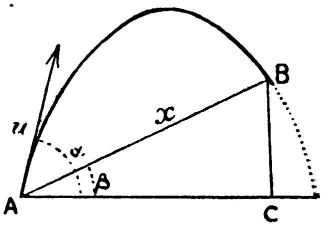


FIG. 116.

horizontal distance AC described in t secs. $= x\cos\beta$,

vertical distance BC described in t secs. $= x\sin\beta$.

Hence $$ut\cos\alpha = x\cos\beta, \quad (1)$$

and $$ut\sin\alpha - \tfrac{1}{2}gt^2 = x\sin\beta. \quad (2)$$

From (1) we have

$$t = \frac{x \cos \beta}{u \cos \alpha}.$$

Substituting this value of t in (2), we have

$$\frac{x \sin \alpha \cos \beta}{\cos \alpha} - \frac{gx^2 \cos^2 \beta}{2u^2 \cos^2 \alpha} = x \sin \beta,$$

or

$$\frac{\sin \alpha \cos \beta}{\cos \alpha} - \frac{gx \cos^2 \beta}{2u^2 \cos^2 \alpha} = \sin \beta;$$

$$\therefore \quad \frac{gx \cos^2 \beta}{2u^2 \cos^2 \alpha} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha}$$

$$= \frac{\sin (\alpha - \beta)}{\cos \alpha};$$

$$\therefore \quad x = u^2 \frac{2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta},$$

or

$$x = u^2 \frac{\sin (2\alpha - \beta) - \sin \beta}{g \cos^2 \beta}.$$

To find the Angle of Elevation which, with a given velocity of projection, will give the Maximum Range on an Inclined Plane.

253. The Range $= u^2 \dfrac{\sin (2\alpha - \beta) - \sin \beta}{g \cos^2 \beta}$,

but $\sin \beta$ and $g \cos^2 \beta$ are constant; hence the range must be a maximum when $\sin (2\alpha - \beta)$ is a maximum; that is, when

$$2\alpha - \beta = 90^\circ,$$

or

$$\alpha = \tfrac{1}{2}(90 + \beta) = 45^\circ + \tfrac{1}{2}\beta;$$

that is, *when the direction of projection bisects the angle between the inclined plane and the vertical.*

Also the maximum range $= u^2 \dfrac{\sin 90 - \sin \beta}{g \cos^2 \beta}$

$$= u^2 \frac{1 - \sin \beta}{g \cos^2 \beta}.$$

EXAMPLES XXXII.

1. A bullet is fired horizontally from a rifle from the top of a tower 100 feet high with a velocity of 1000 feet per second: find how far from the base of the tower will it strike the ground.

2. With what velocity should a bullet be fired horizontally from a loophole in a fort at a height of 81 feet from the ground in order to strike a man at a distance of a mile from the foot of the fort, the ground being supposed to be horizontal?

3. A man fires a bullet from a rifle in a horizontal direction out to sea, and at the same moment he lets a bullet fall to the ground from the same height as the position occupied by the rifle. He sees the splash of the former bullet in the water, and hears the percussion of the latter on the ground: which of these events happens first, and give your reasons?

4. A body is projected with a velocity of 104 *f/s* at an angle of elevation of $\sin^{-1}\frac{5}{13}$: find the horizontal range, the time of flight, and the greatest height reached.

5. A ship is moving with a velocity of 15 miles per hour when a body is let drop from the top of the mast which is 100 feet high: find where it will strike the deck; also find what is the actual velocity and what is the direction of motion of the body in space at the moment when it strikes the deck.

6. The greatest horizontal range of a projectile for a given velocity is 200 feet: find the velocity of projection.

7. If the greatest height reached by a projectile is a quarter the horizontal range, find the angle of elevation.

8. A ball is projected so that initially its vertical component of velocity is 80 feet per second, and its horizontal one 48 feet per second. Where will it be in 3 seconds? What will be its greatest height above the ground? What will be its horizontal range and whole time of flight?

9. Three particles are projected simultaneously from the same point and in the same vertical plane—one vertically upwards, another at an angle of elevation of 30°, and the third horizontally. If their velocities be in the ratio $1:1:\sqrt{3}$, prove that they will always be in a straight line. How does this line move in space?

10. Two particles are projected from the same point and at the same instant with different velocities and at different angles of elevation: prove that the line joining the positions of the particles at any moment moves parallel to itself.

11. A ball is projected with an initial velocity of 156 *f/s* towards the top of a post 60 feet high, whose base is 144 feet from the point of projection: find where and at what angle it will hit the post.

12. With what initial horizontal and vertical components of velocity must a ball be projected so that it will just clear a pole 21 feet high whose base is 36 feet from the point of projection, and have a horizontal range of 120 feet?

13. Two bodies are projected from the same point on a horizontal plane, and with the same velocity at angles of elevation $45^\circ + a$ and $45^\circ - a$, i.e. the angle of elevation of one exceeds 45° by the same amount as that of the other falls short of 45°: prove that the horizontal range in each case is the same.

14. A projectile is fired with a velocity of 800 *f/s* in a direction making an angle $\sin^{-1}\frac{4}{5}$ upwards with the horizontal: find where, and in what direction, it will strike a plane 656 feet below the point of projection.

15. How must a particle be projected so as to pass in a horizontal direction through a point distant 112 feet horizontally from the point of projection, and 49 feet above it?

16. If a ball can just pass through a ring, find the way it must be projected so that it shall pass through the ring when the centre of the ring is distant 240 feet horizontally and 80 feet vertically upwards from the initial position of the centre of the ball and the plane of the ring inclined at $\tan^{-1}\frac{3}{4}$ to the horizon.

Note.—Students reading the book for the first time are advised to omit the remaining examples.

17. Find the focus and directrix of the parabola described by a projectile fired with a velocity of 1000 *f/s* in a direction making an angle $\sin^{-1}\frac{4}{5}$ to the horizon.

18. If particles be projected from the same point at the same instant, with the same velocity, in different directions, prove that they will all at any future instant be on the surface of a sphere.

19. A stone is thrown in such a manner that it would just hit a bird on the top of a tree, and afterwards reach a height double that of the tree; if at the moment of throwing the stone the bird flies away horizontally from the thrower, show that the stone will, notwithstanding, hit the bird if the horizontal velocity of the stone be to that of the bird as $1 + \sqrt{2}$ to 2.

20. A bead slides down a smooth straight wire and then falls freely under the action of gravity; prove that the directrix of its parabolic path is a horizontal line drawn through the upper extremity of the wire.

21. Prove that the time occupied in sliding down all chords drawn from the highest point of a vertical circle is constant. Also show that their subsequent parabolic paths have the same directrix.

22. Show that the greatest range on an inclined plane of 30° is two-thirds of the greatest range on a horizontal plane, the body having in each case the same initial velocity.

23. Find the angle at which a body must be projected in order that the horizontal range may be equal to the distance of the point of projection from the directrix.

24. A body is projected with velocity u, at an inclination α to the horizon, from the foot of a plane inclined at an angle β to the horizon so that its path is in the same plane as the line of greatest slope in the inclined plane; with what velocity should a person rush up the incline in order to be always vertically beneath the body?

25. A body is projected from the foot of a plane inclined at an angle β to the horizontal. At what angle must it be projected in order that it should strike the plane at right angles? Is it necessary in this question to be given the initial velocity?

CHAPTER XVII.

WORK AND ENERGY.

Work.

254. An agent is said to do work when the point of application of the force which it exerts *moves in the direction along which the force acts.* E.g., an engine drawing a train does work; so does a man when he raises a heavy weight off the ground; also, if I push the table at which I write so as to cause it to slide along the floor, I do work; for, in each of these cases, the point of application of the force exerted by the agent moves in the direction along which the force acts.

On the other hand, when there is no movement of the point of application of the force in the direction of the force, no work is being done: a table supporting a heavy weight does no work; neither does a horse do work when pulling steadily in a vain effort to move a heavy wagon which has got fixed in a rut.

255. When the point of application moves in a direction opposite to that of the force, work is said to be done *against* the force, or, in other words, the force is said to do *negative work.*

256. **Definition.**—The work done by a force is measured by *the product of the force, and the distance moved in the direction of the force by its point of application.*

For example, let the force P act on a body {fig. 117 (a)}; then, if its point of application move from A to B a distance s in the direction of P, the work done by P is measured by Ps.

If, however, the point A move through a distance AB {fig. 117 (*b*)}, while the direction of P is at an angle θ to AB, then draw BC perpendicular to the line of action of P, and AC will represent the distance moved by A in the direction of the force.

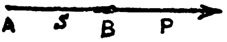


FIG. 117 (*a*).

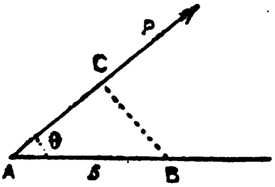


FIG. 117 (*b*).

Hence, the work done by $P = P \times AC = Ps \cos \theta$.

257. **Unit of Work.**—The Gravitation Unit of Work in the F.P.S. system is the work done by a force of one pound weight in moving its point of application in its own direction through a distance of one foot. This unit is called the **Foot-Pound.**

Thus the work done in raising a mass weighing 10 lbs. through a height of 5 feet is 10 × 5, or 50 foot-pounds.

The Absolute Unit of Work is the **Foot-Poundal**, by which is meant the amount of work done by a force of 1 poundal in moving its point of application in its own direction through a distance of 1 foot.

In the C.G.S. system, the absolute unit of work is the **Erg**, which may be defined as the amount of work done by a force of 1 dyne in moving its point of application in its own direction through 1 centimetre.

Ex. 1. Find the work done by gravity when a 56-lb. wt. is let fall through a distance of 10 feet.

[Units of work = 56 × 10 = 560 ft.-lbs.]

Ex. 2. The weight of a train, including the engine, is 100 tons; it is moving at a uniform rate of 30 miles per hour; the resistance due to friction, etc., amounts to 8 lbs. per ton. Find how many units of work are done each second by the engine.

Here the force exerted by the engine is just what is sufficient to overcome the resistances, viz. 100 × 8 or 800 lbs. wt.

Now, 30 miles per hour = 44 feet per second.

Hence the work expended in each second = 800 × 44 = 35,200 ft.-lbs.

Ex. 3. A force of 20 lbs. wt. acts on a body for a certain time, during which the point of application of the force moves a distance of 6 feet in a direction inclined at 60° to the direction of the force: find the work done during that time.

Here the units of work expended $= 20 \times 6 \cos 60 = 20 \times 3 = 60$ ft.-lbs.

Ex. 4. If the pressure of the steam in the cylinder of an engine be 30 lbs. per square inch, the area of the piston 50 square inches, and the length of the stroke 2 feet, find the amount of work done in each stroke.

[Amount of work $= 30 \times 50 \times 2 = 3000$ ft.-lbs. in each stroke.]

258. To prove that *the work done in dragging a body up a smooth inclined plane is equal to the work done in lifting the body through the height of the plane.*

Let W be the weight of the body, and P the force exerted (fig. 74); also, let the height BC be denoted by h, and the length AB by l; then, when P has dragged the body from A to B,

the work done by $P = P \times AB = P \times l$;

but $P = W \sin a$ (Art. 139).

$\therefore$ The work done $= W \sin a \times l$

$$= W \frac{h}{l} \times l = Wh.$$

But, $W \times h =$ the work done in lifting the weight W through the height h. Q.E.D.

259. Similarly, we can prove that if the plane be rough, *the work done in dragging the body up the plane is equal to the work done in dragging the body along a horizontal plane equal in length to the base (supposed to be equally rough) together with the work done in lifting the body through the height of the plane.*

For the work done by $P = P \times l$,

$$= W(\sin a + \mu \cos a)\, l \text{ (Art. 154)}$$

$$= W\left(\frac{h}{l} + \frac{\mu b}{l}\right) l$$

$$= Wh + \mu Wb,$$

where b is the length of the base. But the work done in dragging the body along the horizontal plane = force × distance moved $= \mu Wb$, and the work done in lifting the body a height $h = Wh$. Q.E.D.

260. **The Principle of Work.**—We have seen that by means of the various machines (Art. 88) force may be transmitted from one point to another; but such contrivances can never be a source of energy or work—in fact, work under these circumstances is actually lost owing to friction and the weights of the parts of the machine itself, so that the amount of *work applied* to any machine always exceeds the *work done* by the machine. If this were not the case—that is, if the work done by the power applied were equal to the work done against the weight—then "perpetual motion" would be possible. If, however, we neglect these sources of loss of energy, the following proposition, called *the Principle of Work,* is true of all machines:—

In any machine moving uniformly, provided we neglect friction and the weights of the parts of the machine itself, the work done by the power is equal to the work done against the weight.

Or, taking into account the loss of work, *the work done by the power is equal to the work done against the weight, together with the work done against friction and against the weights of the parts of the machine itself.*

261. This Principle of Work has already been verified in the case of the smooth inclined plane (Art. 251); the work done by the power is the actual work expended in dragging the body up the plane, the work done against the weight being the work that would have to be done in lifting the body through the height of the plane. It may also be verified in the case of the other machines. Take, for example, the First System of Pulleys (Art. 128). Here, supposing there are three movable pulleys, and that the point of application of the power P (fig. 70) is pulled down a vertical height h, this causes each portion of the string supporting the pulley A_3 to be shortened by $\frac{h}{2}$; hence the pulley A_3 is raised through a height $\frac{h}{2}$. Similarly, the pulley A_2 is raised a height $\frac{h}{4}$; also

the pulley A_1 by an amount $\frac{h}{8}$; and, finally, the weight W is raised a vertical distance $\frac{h}{8}$. In this case,

$$W = 2^3P = 8P \text{ (Art. 128)}.$$

Also the work done by the power $= P \times h$, and the work done against the weight

$$= W \times \frac{h}{8} = 8P \times \frac{h}{8} = P \times h:$$

the same amount of work in each case.

Similarly, it may be verified in the case of any of the other systems of pulleys, the different orders of lever, or the wheel and axle.

262. The Principle of Work, in the case of frictionless machines, may be stated in another form, viz.:—

What is gained in power is lost in despatch. This principle can be applied to obtain the relation between the forces acting at different parts of even the most complicated machines by comparing the distances moved over by the points of application of these forces during the same intervals of time.

263. *To find the relation between the Power and the Weight in the Screw by means of the Principle of Work.*

Let r = length of power-arm.

Let c = circumference of circle described by the extremity of the power-arm.

Now, work done by P in one revolution

$$= P \times \text{distance moved} = P \times c;$$

and work done against W in one revolution

$$= W \times \text{vertical distance moved} = W \times i,$$

where i is the pitch of the screw.

But work done by power = work done against the weight;

$$\therefore \quad P \times c = W \times i, \quad \text{or} \quad P \times 2\pi r = W \times i.$$

Q. E. D.

264. **Theorem.**—*If a number of weights are raised through different heights, the total amount of work done is equal to the amount which would have to be expended in raising a weight equal to the sum of the weights through the height through which the centre of gravity of the weights has been raised.*

Let $w_1, w_2, w_3 \ldots$ be the weights,
and $x_1, x_2, x_3 \ldots$ be the respective heights of their centres of gravity above a given plane;

then, if h be the height of their common centre of gravity above the same plane, we have (Art. 76)

$$(w_1 + w_2 + w_3 + \ldots)\, h = w_1x_1 + w_2x_2 + w_3x_3 + \ldots \quad (1)$$

Now, let the weights be lifted so that their centres of gravity are at heights $k_1, k_2, k_3 \ldots$ respectively above the given plane; and let their common centre of gravity be now at a height k above the same plane. Then

$$(w_1 + w_2 + w_3 + \ldots)\, k = w_1k_1 + w_2k_2 + w_3k_3 + \ldots \quad (2)$$

Subtracting (1) from (2) we have

$$(w_1 + w_2 + w_3 + \ldots)(k - h) = w_1(k_1 - x_1) + w_2(k_2 - x_2) + w_3(k_3 - x_3) + \ldots$$

But the right-hand side of this equation measures the total amount of work done in lifting the several weights; and the left-hand side the amount of work done in lifting the sum of the weights through the distance through which their common centre of gravity has been raised.

265. **Definition.**—The **Power** of an agent is the rate at which it can do work—i.e. the number of units of work which it can perform in one unit of time.

266. **Definition.**—The *Unit of Power* generally adopted is **One Horse-Power**, which may be defined as the power of doing 33,000 foot-pounds of work per minute, or 550 foot-pounds of work per second.

We shall use the letters H.P. to denote the horse-power of an engine.

Thus, a motor-car of 50 horse-power is capable of doing 50 × 550 or 27,500 foot-pounds of work each second.

Ex. 1. Find the work done in emptying a shaft filled with water, the depth being 200 feet, and the diameter of the shaft, which is circular in cross-section, 14 feet. Also find the H.P. of the engine which would empty the shaft in 4 hours, taking the weight of a cubic foot of water to be 1000 ozs.

Evidently, in emptying the water out of the shaft, the height through which the centre of gravity of the whole mass of water is raised is 100 feet (half the depth of the shaft).

The volume of the water

$$= \pi \times (7)^2 \times 200 = \frac{22}{7} \times 49 \times 200 = 30800 \text{ cubic feet.}$$

Hence, the weight of water

$$= 30800 \times \frac{1000}{16} \text{ lbs.} = 1925000 \text{ lbs.}$$

$$\therefore \text{ The work done} = \text{weight} \times \text{height through which the C.G. is raised}$$
$$= 192500000 \text{ ft.-lbs.}$$

Again, to find the H.P. of the engine:

$$\text{The work done in 4 hours} = 192500000 \text{ ft.-lbs.:}$$

$$\therefore \text{ Work in 1 second} = \frac{192500000}{4 \times 60 \times 60} \text{ ft.-lbs.;}$$

$$\therefore \text{ H.P. of engine} = \frac{192500000}{4 \times 60 \times 60 \times 550} = 24{\cdot}3 \text{ nearly.}$$

✓ Ex. 2. Find the H.P. of an engine which draws a train of mass 200 tons with a uniform velocity of 30 miles per hour along horizontal rails, the resistance due to friction, etc., being 10 lbs. weight per ton.

Here the resistance $= 200 \times 10 = 2000$ lbs. wt.

Hence, work done per second $= 2000 \times 44$ ft.-lbs.

$$\therefore \text{ H.P. of engine} = \frac{2000 \times 44}{550} = 160.$$

Ex. 3. Find the H.P. at which an engine must be able to work in order, by the exertion of a constant force, to generate in 10 minutes a velocity of 30 miles per hour up an incline of 1 in 400 in a train of mass 200 tons, taking the resistance due to friction to be 12 lbs. wt. per ton.

Since in 10 minutes a velocity of $44 f/s$ is generated,

$$\therefore \text{ the acceleration of train} = \frac{44}{600} = \frac{11}{150} f/s/s.$$

Let P poundals be the force exerted by the engine.

The resistance due to friction, etc.,

$$= 200 \times 12 \text{ lbs. wt.} = 2400\, g \text{ poundals.}$$

The resolved part of the weight along plane

$$= 200 \times 2240 \times \frac{1}{400}\, g \text{ poundals} = 1120\, g \text{ poundals.}$$

Hence, $P - 2400g - 1120g = 200 \times 2240 \times \frac{11}{150}$;

$$\therefore \quad P = 3520 \times 32 + 200 \times 2240 \times \frac{11}{150} \text{ poundals,}$$

$$= 32 \times 11 \times 10 \left(32 + \frac{28}{3}\right) = \frac{32 \times 11 \times 10 \times 124}{3} \text{ poundals,}$$

or $$P = \frac{11 \times 10 \times 124}{3} \text{ lbs. wt.}$$

Hence, when the train has reached a speed of 30 miles per hour,

$$\text{the work done per second} = \frac{11 \times 10 \times 124 \times 44}{3} \text{ ft. lbs.}$$

Hence, the H.P. of engine $= \frac{11 \times 10 \times 124 \times 44}{3 \times 550} = 363\frac{11}{15}$.

EXAMPLES XXXIII.

1. Find the amount of work done in raising a mass of 1 cwt. through a height of 20 feet.

2. Find the amount of work done per hour by an engine in dragging a truck whose mass is 5 tons along horizontal rails with a uniform velocity of 20 feet per second, the resistance due to friction, etc., being 15 lbs. wt. per ton.

3. Find the uniform velocity with which an engine of 40 H.P. will draw a train of mass 80 tons along a level railroad, the resistance due to friction, etc., being 10 lbs. wt. per ton.

4. Find the work done in raising stones necessary to build a column 100 feet high, the section being uniform and 12 feet square, a cubic foot of stone weighing 140 lbs.

5. The height of a smooth inclined plane is to its base as 5 to 12; the length of the slope is 26 feet: find the work done in dragging a mass of 52 lbs. up the plane, the force being parallel to the plane. Also prove that the amount of work is equal to that expended in raising the mass through a height equal to the height of the plane.

6. In example 5 find the amount of work done in dragging the mass up the plane by means of a horizontal force.

7. In how many hours would an engine of 30 H.P. empty a vertical shaft full of water, if the diameter of the shaft be 7 feet, and the depth be 300 feet?

$$\left[\pi = \frac{22}{7}, \text{ and the weight of a cubic foot of water} = 1000 \text{ ozs.}\right]$$

8. Find the H.P. of an engine which will drag a train of mass 400 tons up a smooth incline of 1 in 200 at a uniform speed of 15 miles per hour.

9. If earth weighing 120 lbs. per cubic foot is in a bank 600 feet long, whose section is a trapezium, 14 feet high, 4 feet broad on top, and 10 feet broad below, what work must be done in order to form it into a bank, whose section is a triangle 9 feet high by 16 broad, on a plane 40 feet higher than the base of the original bank?

10. Find the H.P. at which an engine must be able to work so that by exerting a constant force it shall pull a train of mass 250 tons half a mile up a slope of 1 in 200 in 5 minutes from rest, taking the resistance due to friction to be 10 lbs. wt. per ton.

11. Prove that the work done by the resultant of two concurrent forces in moving a particle in any direction is equal to the algebraic sum of the work done by the component forces.

This follows at once from the fact that the resolved part of the resultant in the direction in which the particle moves is equal to the algebraic sum of the resolved parts of the forces (Art. 34).

On multiplying by the distance moved by the particle the above result follows (Art. 249).

267. **Definition: Impulse.**—The *Impulse* of a force in a given time is *the product of the intensity of the force* if it be constant, or *its mean intensity if it be variable, and the time during which it acts.*

Thus the impulse of a force P, acting for t units of time, is $P.t$.

268. *To prove that the Impulse of a force in a given time is equal to the Change in Momentum of the particle on which the force acts during that time.*

Let P be a constant force (expressed in absolute units), and let it act on a particle of mass m for t seconds during which the velocity of the particle is changed from u to v.

Now, the impulse of P during t seconds $= Pt = mft$: but

$$v = u + ft, \quad \therefore ft = v - u;$$

and, substituting this value of ft, we have

$$Pt = m(v - u).$$

But $m(v - u) =$ the change of momentum; therefore, etc.

Corollary.—If the body start from rest, then $u = 0$, and the equation becomes $Pt = mv$.

269. **Newton's Second Law.**—In the statement of the Second Law of Motion by "*impressed force*" Newton means *impulse*; hence this law might be stated thus:—*The change of momentum of a body in a given time is numerically equal to the impulse which produces it, and is in the same direction.*

270. **Impulsive Forces.**—Forces of great intensity which act for indefinitely small intervals of time, and then cease, are called *impulsive forces.* A cricket-ball, when struck with a bat, a nail by a hammer, and a shot projected from a gun, are examples of this kind of force. Under these circumstances, the impulse is still measured by Pt; but, P being very great, and the interval of time t being very small, they cannot be separated, and therefore cannot be measured, as in the case of finite forces acting for definite intervals of time. Impulsive forces are therefore compared by measuring the momenta they generate irrespective of the time.

271. **Action and Reaction.**—In the case of a gun firing a shot, the forward force exerted by the explosion on the shot is at any instant equal and opposite to the backward force exerted on the gun; that is, the impulses on the shot and gun are equal and opposite, and therefore the momentum generated in each is the same. Hence, if

m = mass of gun, and v = the velocity of recoil,

m' = mass of shot, and v' = velocity of shot,

we have $$mv = m'v'.$$

Ex. 1. Find the impulse necessary to produce a velocity of 20 feet per second in a mass of 10 lbs. Also, if the time be 5 seconds, find the magnitude of the force.

Here $Pt = mv = 10 \times 20 = 200$ units of impulse.

Also, $P \cdot 5 = 200$; $\therefore$ $P = 40$ poundals $= 1\frac{1}{4}$ lbs. wt.

Ex. 2. A ball of mass $\frac{1}{2}$ lb., moving with a velocity of 80 feet per second, is struck by a bat, owing to which its velocity is reversed in direction and equal to 60 feet per second; if the time during which the blow lasts be $\frac{1}{200}$th of a second, find the average intensity of the force during that interval.

Here the change in velocity $= 80 + 60 = 140$;

therefore the change in momentum $= \frac{1}{2} \times 140 = 70$.

Hence $P \times \frac{1}{200} = 70$; $\therefore$ $P = 14{,}000$ poundals $= 437\frac{1}{2}$ lbs. wt.

Ex. 3. A shot of mass $\frac{1}{2}$ oz. is projected from a gun whose mass is 12 lbs. with a velocity of 1600 feet per second: find the velocity of recoil of the gun.

Let v = velocity of recoil of gun;

$$\therefore \quad 12v = \frac{1}{2 \times 16} \times 1600 = 50;$$

$$\therefore \quad v = 4\tfrac{1}{6} \text{ feet per second.}$$

Energy.

272. **Definition.**—*Energy* is *capacity for doing work.*

Energy is of two kinds, viz. *Potential Energy* and *Kinetic Energy.*

273. **Definition.**—The *Potential Energy* of a body is *its capacity for doing work in consequence of its position or configuration as compared with some standard position or configuration.*

The water stored up in a mill-dam at a height above that of the water-wheel has potential energy.

Potential energy is stored up in the weights of a clock on being wound up, as also in the mainspring of a watch when it is wound up.

When air or any gas is compressed, potential energy is stored up in it.

In each of these cases, work can be performed by gradually using up the potential energy of the body or bodies in which that energy has been stored.

274. **Potential Energy Relative, not Absolute.**—It is evident from the above definition that when we speak of the potential energy of a body, we mean its energy with reference to some standard position which may be regarded as the zero position. Thus, if the weights of a "grandfather" clock when wound up are at a height of 5 feet above the floor on which the clock stands, they have a certain amount of potential energy with reference to the level of the floor, which level, in this case, is the zero position. If, however, openings be made in this floor large enough for the weights to pass through, and if the strings on which the weights are suspended are long enough to allow the weights to descend to a second floor—let us say, 15 feet below the previous one—the potential energy possessed by the weights in this case is that due to a height of 20 feet, i.e. 4 times as great as it was with reference to the first floor. It follows from this that the difference between the potential energy of a mass m, placed at a height h above that of an equal mass, is the work that would have to be done in raising m through a height h, viz. mgh.

275. **Definition.**—The *Kinetic Energy* of a body is *its capacity for doing work in consequence of its motion.*

The water in a mill-dam in falling on the mill-wheel has lost its potential energy, but it has gained kinetic energy. Also, the mass of a hammer can do work in consequence of its motion, as when used for driving a nail into wood.

276. It is easy to see that *the kinetic energy of a particle* is measured by *half the product of its mass and the square of its velocity.*

For, if m be the mass of the particle and u its velocity, and P a constant force which, resisting its motion, is able to reduce it to rest after it has passed over a space s, then the work done before coming to rest $= Ps = mfs$; but

$$0 = u^2 - 2fs;$$

$$\therefore \quad fs = \tfrac{1}{2}u^2; \quad \therefore \quad mfs = \tfrac{1}{2}mu^2.$$

Hence $\frac{1}{2}mu^2$ represents the work done, and is therefore equal to the kinetic energy initially possessed by the particle.

277. **Theorem.**—To show that the change in the kinetic energy of a particle is equal to the work done on the particle.

Let a mass m change its velocity from u to v, owing to the action of a constant force P while the particle moves through a space s.

Then the work done $= Ps = mfs$;

but $\qquad v^2 = u^2 + 2fs; \quad \therefore \quad fs = \frac{1}{2}v^2 - \frac{1}{2}u^2,$

and $\qquad \therefore \quad mfs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}m(v^2 - u^2);$

but $\frac{1}{2}m(v^2 - u^2) =$ the change in the kinetic energy; $\therefore$ etc.

To prove that, if a body fall from a height h, the sum of its potential and kinetic energies at each instant during its fall is constant, and is equal to the potential energy it possessed at starting.

278. For, if at any instant it has fallen a height s, then $v^2 = 2gs$; hence its kinetic energy

$$= \tfrac{1}{2}mv^2 = \tfrac{1}{2}m \,.\, 2gs = mgs.$$

But at this instant it is at a height $h - s$ above the ground, and therefore its potential energy $= mg(h - s)$.

Hence, the sum of kinetic and potential energies

$$= mgs + mg(h - s) = mgh;$$

but mgh is the potential energy it possessed at starting; therefore, etc.

Corollary.—From this it follows that the loss of potential energy is equal to the gain in kinetic energy, which is an example of the **Principle of the Conservation of Energy**, which states that energy may be transformed from one kind of energy to another, but can never be destroyed, or, in other words, the total amount of energy in the Universe is constant.

Ex. A body of mass 10 lbs. is let fall through a height of 100 feet: find its kinetic energy when it reaches the ground, and show that it is equal to the potential energy stored up in the body before starting.

$$v^2 = 2 \times 32 \times 100; \quad \therefore \quad v = 80 f/s.$$

$$\text{Kinetic energy} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}10 \times 80^2 \text{ foot-poundals}$$
$$= 32000 \text{ foot-poundals}$$
$$= 1000 \text{ foot-pounds};$$

$$\text{but potential energy at starting} = 10 \times 32 \times 100 \text{ foot-poundals}$$
$$= 1000 \text{ foot-pounds}.$$

EXAMPLES XXXIV.

1. A truck, whose mass is 5 tons, is moving along horizontal rails with a velocity of 8 feet per second, and is struck from behind by an engine which causes it to change its velocity to one of 12 feet per second: if the duration of the blow be $\frac{1}{20}$th of a second, find the impulse and the average intensity of the force during that interval.

2. A shot of mass 100 lbs. is projected from a gun whose mass is 20 tons with a velocity of 2000 feet per second: find the velocity with which the gun recoils.

3. A shot of mass 20 lbs. is projected from a gun whose mass is 5 tons with a velocity of 1600 feet per second: find the velocity of recoil. Also, if the gun be on a horizontal plane, and if its recoil be resisted by a constant force of 25 cwt., find how far it will recoil before coming to rest.

4. A body of mass 6 lbs. falls from the top of a tower 25 feet high: find its kinetic and potential energies (1) when it has fallen a height of 9 feet, (2) when it has fallen 16 feet, (3) when it reaches the ground; and show in each case the sum of the two forms of energy is a constant quantity.

5. A body of mass 10 lbs. is projected vertically upwards with a velocity of 160 feet per second: find its kinetic and potential energies (1) after 1 second, (2) after 5 seconds; and show that in each case the total energy is the same.

6. A man is placed in the middle of a room, the floor of which is perfectly smooth: show how, by an application of the principle of action and reaction being equal and opposite, he could reach the door.

CHAPTER XVIII.

IMPACT.

279. When one mass meets another mass with a different velocity, they are said to *impinge*; and the collision of the two masses is called an *impact*.

When two bodies, such as two billiard-balls, moving in the same or opposite directions, collide, the interval of time during which they are in contact is very small, and, during this interval, the pressure between the two bodies is not constant, as it increases from zero at the beginning of the interval to a maximum, and then decreases to zero at the moment of separation.

280. By the Third Law of Motion equal and opposite impulses are imparted to the bodies; and therefore the momentum generated in one must be equal and opposite to that generated in the other; and hence the sum of the momenta of the two bodies is unaltered—i.e. *the algebraic sum of the momenta before impact is equal to the algebraic sum of the momenta after.*

It is to be borne in mind that "algebraic sum" means the sum of the momenta, taking into account the direction of the motion—i.e. whether this direction is positive or negative; and hence the above statement applies whether the bodies are moving in the same or in opposite directions.

281. **Definition.**—Two bodies are said to be *inelastic* when, after direct impact, they move with a common velocity.

Thus, if two lumps of putty stick together after collision, and therefore move with a common velocity, they are said to be inelastic.

282. *A mass m moving with velocity u overtakes a mass m′ moving in the same direction with velocity u′: to find their common velocity after impact if both stick together.*

Let v = their common velocity after impact.

Now, the Sum of Momenta before impact = the Sum of Momenta after.

$$\therefore \quad mu + m'u' = (m + m')v; \quad \therefore \quad v = \frac{mu + m'u'}{m + m'}.$$

If the bodies *meet*—i.e. if they move in opposite directions—then, regarding u as positive in direction, u' must be taken as negative.

Ex. 1. A mass of 10 lbs. moving with a velocity of 8 feet per second overtakes a mass of 5 lbs. moving in the same direction with a velocity of 3 feet per second: if the bodies remain together after impact, find their common velocity and the kinetic energy lost.

Let v be the common velocity after impact; then

the Sum of Momenta before = the Sum of Momenta after;

or, $$10 \times 8 + 5 \times 3 = (10 + 5)v, \quad 95 = 15v;$$

$$\therefore \quad v = 6\tfrac{1}{3} f/s.$$

Also, the total energy before

$$= \tfrac{1}{2}10 \times 8^2 + \tfrac{1}{2}5 \times 3^2 = 342{\cdot}5 \text{ foot-poundals},$$

and the total energy after

$$= \tfrac{1}{2}15 \times \left(\frac{19}{3}\right)^2 = 300{\cdot}8 \text{ foot-poundals}.$$

Hence the energy lost = 41·7 foot-poundals.

Ex. 2. If the masses in last question meet, find the common velocity and the loss of kinetic energy.

Here, $$10 \times 8 - 5 \times 3 = (10 + 5)v, \quad \text{or} \quad 65 = 15v;$$

$$\therefore \quad v = 4\tfrac{1}{3} f/s.$$

Also, total energy after impact $= \tfrac{1}{2}15 \left(\frac{13}{3}\right)^2 = 140{\cdot}8$ **foot-poundals.**

Hence, energy lost = 342·5 − 140·8 = 201·7 foot-poundals.

283. **Definition.**—Two masses are said to be *elastic* when, after direct impact, their velocities are not the same.

Direct Impact upon a Plane at Rest.—If a sphere moving with velocity u strike a plane which is at rest at right angles—as, for instance, when a golf-ball is dropped on the floor of a room—it has been proved by experiment that, if v be the velocity of recoil, then $v = eu$, where e is a constant quantity depending on the substances of which the body and plane are composed. This quantity e is called the *coefficient of restitution or elasticity* of the substances, which may therefore be defined as *the ratio of the velocity of recoil to the velocity of approach.*

Thus, if a ball fall on the ground with a velocity of 20 feet per second, and rebound with a velocity of 10 feet per second, then the coefficient of restitution $e = \frac{10}{20} = \frac{1}{2}$.

Most bodies possess the power of regaining their original form after that form has been disturbed by collision with some other body; this property, which is called elasticity, being the cause of the recoil after the collision. According to their elasticity bodies are divided into—

1. Perfectly elastic where $v = u$ or $e = 1$.
2. Imperfectly elastic where v is $< u$ or e is < 1.
3. Perfectly inelastic where $v = 0$ or $e = 0$.

Ex. A ball falls from a height of 100 feet on a horizontal plane: find the heights to which it rises after the first and second rebounds respectively, the coefficient of restitution being $\frac{1}{2}$.

Here $v^2 = u^2 + 2fs$ or $v^2 = 2 \times 32 \times 100$;

$$\therefore \quad v = 80 f/s.$$

Hence, the velocity of recoil $= \frac{1}{2} 80 = 40$.

Now the body is projected up with a velocity of $40 f/s$; it is required to find the height to which it ascends:

$$0 = 40^2 - 2 \times 32s; \quad \therefore \quad s = 25 \text{ feet}.$$

Similarly it may be proved that the height to which it rebounds after the second recoil $= 6\frac{1}{4}$ feet.

284. **Direct Impact of Two Spheres.**—*A body whose mass is m, moving with a velocity u, overtakes a body of mass m′ moving in the same direction with velocity u′: to find their velocities after impact.*

In this case—

The relative velocity of recoil $= e \times$ the relative velocity of approach; and also the body which has the lesser velocity before has the greater velocity after impact. Hence, v and v' being the respective velocities after impact,

$$v - v' = - e(u - u').$$

Also, the sum of the momenta before impact is equal to the sum of momenta after;

$$\therefore \quad mu + m'u' = mv + m'v',$$

and solving for v and v' we obtain

$$v = \frac{mu + m'u' - em'(u - u')}{m + m'},$$

and
$$v' = \frac{mu + m'u' + em(u - u')}{m + m'}.$$

If the two bodies are moving in opposite directions, then, choosing any one velocity to be positive, the other is to be considered negative.

285. **Loss of Kinetic Energy in Impact.**—*To find an expression for the loss of kinetic energy in direct impact of two spheres in terms of the masses and original velocities.*

The total kinetic energy before impact $= \frac{1}{2}mu^2 + \frac{1}{2}m'u'^2$, and the total kinetic energy after impact $= \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2$.

Then, if we denote the loss by K, we have

$$K = \tfrac{1}{2}(mu^2 + m'u'^2) - \tfrac{1}{2}(mv^2 + m'v'^2).$$

But we have the two equations (Art. 284)

$$mv + m'v' = mu + m'u', \qquad (1)$$
$$v - v' = -e(u - u'). \qquad (2)$$

Squaring (1) and (2), we have

$$m^2v^2 + 2mm'vv' + m'^2v'^2 = m^2u^2 + 2mm'uu' + m'^2u'^2$$

and
$$v^2 - 2vv' + v'^2 = e^2(u^2 - 2uu' + u'^2).$$

Multiplying the second of these equations by mm', and adding it to the first, we have

$$(m + m')(mv^2 + m'v'^2) = m^2u^2 + 2mm'uu' + m'^2u'^2 + e^2mm'(u^2 - 2uu' + u'^2).$$

Hence

$$\tfrac{1}{2}(mv^2 + m'v'^2) = \frac{m^2u^2 + 2mm'uu' + m'^2u'^2 + e^2mm'(u^2 - 2uu' + u'^2)}{2(m + m')}.$$

Subtracting this result from $\frac{1}{2}(mu^2 + m'u'^2)$ we finally obtain that

$$K = \frac{1 - e^2}{2}\,\frac{mm'}{m + m'}(u - u')^2.$$

From this result it follows that, since e cannot be greater than unity, $1 - e^2$ cannot be negative; hence there is always a loss of kinetic energy after impact, except when $e = 1$, when the total kinetic energy remains unaltered.

286. *To find the total impulse between two spheres in direct impact.*—Let I denote the impulse given by m to m'. This must equal the change in momentum of m'.

Hence $I = m'(v' - u')$.

Substitute the value of v' (Art. 284);

$$\therefore \quad I = m'\left\{\frac{mu + m'u' + em(u - u')}{m + m'} - u'\right\}$$

$$= m' . \frac{mu + m'u' + em(u - u') - mu - m'u'}{m + m'}$$

$$= m' . \frac{m(u - u') + em(u - u')}{m + m'},$$

or
$$I = \frac{mm'(1 + e)(u - u')}{m + m'}.$$

The impulse given by m' to m is evidently equal and opposite to that given by m to m'.

287. When one sphere impinges directly on another the following may be regarded as the nature of the mutual action. The whole duration of impact may be divided into two parts: the first, a period of *compression*, during which the centres of the two spheres are approaching one another; the second, a period of *restitution*, during which each sphere tends to recover its original shape from its previous condition of compression; hence, during this second period the centres are receding from one another. Also, at the period of *greatest compression*, at the end of the first period *the relative velocity is zero*, or, in other words, both spheres are moving with a common velocity. The distinct separation of these two periods of compression and restitution can be actually observed in the case of a collision between two railway carriages, fitted with spring buffers, as well as the instant of greatest compression. Again, if the spheres were perfectly inelastic, there would be no period of restitution, but they would continue to move after impact with a common velocity. Hence the common velocity of *any* two spheres at the moment of greatest compression is equal to what their

common velocity would be after impact if they were perfectly inelastic, i.e. if e were $= 0$. This velocity is given by the equation

$$v = \frac{mu + m'u'}{m + m'}. \quad \text{(Art. 282.)}$$

288. *To find the magnitudes of the impulses during the two periods of compression and restitution.*

Let the magnitudes of the impulses during compression and restitution be represented by I_1 and I_2 respectively. Then the amount of impulse received by either mass during the first period is measured by the change in momentum of that mass. As it is immaterial which mass we take, let us take m'.

Hence

$$I_1 = m'\left(\frac{mu + m'u'}{m + m'} - u'\right) = \frac{mm'(u - u')}{m + m'}.$$

Similarly we have

$$I_2 = m'\left(v' - \frac{mu + m'u'}{m + m'}\right)$$

$$= m'\left(\frac{mu + m'u' + em(u - u')}{m + m'} - \frac{mu + m'u'}{m + m'}\right)$$

$$= \frac{emm'(u - u')}{m + m'}.$$

Hence we conclude that

$$I_2 = eI_1.$$

By adding the two expressions for I_1 and I_2 the student can verify that

$$I = I_1 + I_2.$$

These results can also be obtained by making $e = 0$ in the value of I (Art. 286), which gives the value of I_1, and on subtracting the value of I_1 from I we obtain the value of I_2.

289. *To find the changes in the kinetic energy during the periods of compression and restitution.* During the period of compression there is a loss of kinetic energy followed by a gain during the period of restitution. Let this loss and gain be denoted by K_1 and K_2 respectively. Now, evidently

$$K_1 = \tfrac{1}{2} mu^2 + \tfrac{1}{2} m'u'^2 - \tfrac{1}{2}(m + m')\left(\frac{mu + m'u'}{m + m'}\right)^2$$

$$= \tfrac{1}{2}\left\{mu^2 + m'u'^2 - \frac{m^2u^2 + 2mm'uu' + m'^2u'^2}{m + m'}\right\}$$

$$= \tfrac{1}{2}\,\frac{mm'(u - u')^2}{m + m'}.$$

This value for K_1 can also be obtained by making $e = 0$ in the value for K (Art. 285).

Similarly we can find the value of K_2, for

$$K_2 = \tfrac{1}{2} mv^2 + \tfrac{1}{2} m'v'^2 - \tfrac{1}{2}(m + m')\left(\frac{mu + m'u'}{m + m'}\right)^2,$$

but, as the algebraic work involved in obtaining the final result would be tedious, we can obtain it more rapidly thus:

Evidently the net loss $K = K_1 - K_2$;

$$\therefore\ K_2 = K_1 - K$$

$$= \tfrac{1}{2}\,\frac{mm'(u - u')^2}{m + m'} - \frac{1 - e^2}{2}\,\frac{mm'(u - u')^2}{m + m'}$$

$$= \tfrac{1}{2}e^2 . \frac{mm'(u - u')^2}{m + m'}.$$

Hence $K_2 = e^2K_1$, or gain during restitution $= e^2 \times$ loss during compression.

Note.—The student is advised in working examples to work from first principles, and not to make use of the above formulae.

Ex. 1. A mass of 10 lbs. with a velocity of 8 feet per second overtakes a mass of 6 lbs. moving in the same direction with a velocity of 4 feet per second: find their velocities after impact, the coefficient of restitution being $\tfrac{1}{2}$.

Here $\qquad v - v' = -\tfrac{1}{2}(8 - 4) = -2;$

also $\qquad 10v + 6v' = 10 \times 8 + 6 \times 4 = 104;$

$$\therefore\ v = 5\tfrac{3}{4} f/s \text{ and } v' = 7\tfrac{3}{4} f/s.$$

Ex. 2. With what velocity must a sphere of mass 20 lbs. be projected against one of 40 lbs., so that, taking the coefficient of restitution to be $\frac{1}{2}$, the velocity of the latter shall be changed from a forward one of 30 *f/s* to a backward one of 30 *f/s*? Find also the energy lost in foot-pounds.

Here, taking the direction of the velocity u of the 20 lbs. mass as positive,

$$v - 30 = -\tfrac{1}{2}(u + 30),$$

and $$20u - 40 \times 30 = 20v + 40 \times 30;$$

and solving, we obtain $u = 90$ and $v = -30$ *f/s*.

Also, energy before impact $= \frac{1}{2}\,20 \times 90^2 + \frac{1}{2}\,40(30)^2 = 99000$ foot-poundals,

and energy after impact $= \frac{1}{2}\,20\,(30)^2 + \frac{1}{2}\,40\,(30)^2 = 27000$ foot-poundals;

hence, energy lost $= 72000$ foot-poundals

$= 2250$ foot-pounds.

Ex. 3. A mass of 5 lbs., moving with a velocity of 14 *f/s*, impinges on a mass of 3 lbs. moving in the same direction with a velocity of 8 *f/s*; taking the coefficient of restitution to be $\frac{1}{3}$, find (*a*) the velocity of each after impact, (*b*) the total kinetic energy lost, (*c*) the total impulse, (*d*) the impulses during the periods of compression and restitution respectively, (*e*) the loss of kinetic energy during compression, and the gain during restitution.

(*a*) Here $$v - v' = -\tfrac{1}{3}(14 - 8) = -2,$$

and $$5v + 3v' = 5 \times 14 + 3 \times 8 = 94.$$

Solving for v and v', we obtain

$$v = 11 \quad \text{and} \quad v' = 13.$$

(*b*) Also energy lost $= \frac{1}{2}5 \times 14^2 + \frac{1}{2}3 \times 8^2 - (\frac{1}{2}5 \times 11^2 + \frac{1}{2}3 \times 13^2)$

$= 30$ foot-poundals.

(*c*) The total impulse I = the change in momentum in either mass. Hence, taking the mass of 3 lbs., we have

$$I = 3\,(13 - 8) = 15.$$

(*d*) The common velocity with which both are moving during the instant of greatest compression $= \dfrac{mu + m'u'}{m + m'} = \dfrac{70 + 24}{8} = 11\frac{3}{4}$ *f/s*. Hence, taking the change in momentum of the 3 lbs. mass, we have

the impulse during compression $I_1 = 3\,(11\frac{3}{4} - 8) = 11\frac{1}{4}$,

and the impulse during restitution $I_2 = 3\,(13 - 11\frac{3}{4}) = 3\frac{3}{4}$.

(*e*) Loss of energy during compression

$$K_1 = \tfrac{1}{2}5 \times 14^2 + \tfrac{1}{2}3 \times 8^2 - \tfrac{1}{2}(5 + 3)\,(11\tfrac{3}{4})^2$$
$$= 33\tfrac{3}{4} \text{ foot-poundals,}$$

and gain of energy during restitution $K_2 = \frac{1}{2}5 \times 11^2 + \frac{1}{2}3 \times 13^2 - \frac{1}{2}\,(5 + 3)(11\frac{3}{4})^2$

$= 3\frac{3}{4}$ foot-poundals.

From the above results the student can verify that

$$K = K_1 + K_2; \quad I = I_1 + I_2; \quad I_2 = eI_1; \text{ and } \quad K_2 = e^2K_1.$$

Ex. 4. If two masses impinge directly on one another, find the conditions in order that they may interchange velocities.

Using the same notation as before, we require that $v = u'$, and $v' = u$. Hence (Art. 284)

$$mu' + m'u = mu + m'u',$$

or
$$(m - m')(u - u') = 0.$$

Hence $m - m' = 0$, or $m = m'$ (for $u - u'$ cannot $= 0$).

Also
$$u' - u = -e(u - u'),$$

or
$$(1 - e)(u - u') = 0,$$

or
$$1 - e = 0; \quad \therefore \quad e = 1.$$

Hence the two masses must be equal and perfectly elastic.

Ex. 5. A number of balls A, B, C, &c., formed of the same substance are placed in a straight line at rest. A is then projected with a velocity u so as to impinge on B; then B impinges on C; and so on. If the mass of A be m, find the masses of B, C, &c., so that each of the bodies A, B, C, &c., may be at rest after impinging on the next; and find the velocity of the nth ball just after it has been struck by the $(n - 1)$th ball.

Here, if the mass of B be m', we have

$$mv + m'v' = mu + m'u';$$

but
$$v = 0 \quad \text{and} \quad u' = 0,$$
$$\therefore \quad m'v' = mu.$$

Also
$$v - v' = -e(u - u'),$$

or
$$-v' = -eu, \quad \text{or} \quad v' = eu.$$

Substituting this value of v' in the first result, we have

$$\therefore \quad m'eu = mu; \quad \therefore \quad m' = \frac{m}{e}.$$

Similarly the mass of C $= \dfrac{\text{mass of } B}{e} = \dfrac{m}{e^2}$,

and the mass of the nth ball $= \dfrac{m}{e^{n-1}}$.

Again, the velocity of B or v' $= eu$;
similarly the velocity of C $= e^2u$,
and the velocity of the nth ball $= e^{n-1}u$.

290. **Oblique Impact. Definition.**—An impact which is not *direct* is said to be *oblique*.

Oblique Impact of a Sphere on a Smooth Plane.

Fig. 118 represents the sphere at the moment of impact with the plane *MN*. *XY* is a parallel to *MN* through the centre of the sphere. Let u and v be the velocities of the sphere before and after impact, and let their directions make angles α and β respectively with *MN* or *XY*. We are given u and α, and it is required to find v and β.

Erect a perpendicular OZ to XY or MN. Then, since the plane MN is smooth, the direction of the impulsive force due to the impact is along OZ, perpendicular to MN, and therefore does not alter the resolved part of the velocity of the sphere along MN or XY. Also the resolved part of the velocity of the sphere perpendicular to MN, *i.e.* along OZ, must follow the same rule as in the case of direct impact. Hence we have the two rules:

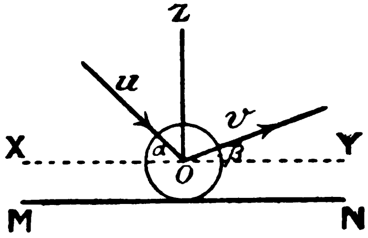


Fig. 118.

(1) *The resolved part of the velocity of the sphere parallel to the plane after impact is equal to the resolved part of the velocity parallel to the plane before impact.*

(2) *The resolved part of the velocity perpendicular to the plane after impact* = e × *resolved part of the velocity perpendicular to the plane before impact, or, in other words, the resolved part of the velocity of recoil perpendicular to the plane* = e × *the resolved part of the velocity of approach perpendicular to the plane.*

From the first of these rules we have the equation

$$v \cos \beta = u \cos a, \qquad (1)$$

and from the second

$$v \sin \beta = eu \sin a. \qquad (2)$$

On squaring these two equations and adding we obtain

$$v^2(\sin^2\beta + \cos^2\beta) = u^2(\cos^2 a + e^2 \sin^2 a),$$

but

$$\sin^2\beta + \cos^2\beta = 1;$$

$$\therefore \quad v^2 = u^2(\cos^2 a + e^2 \sin^2 a).$$

Again, on dividing (2) by (1), we obtain

$$\frac{v \sin \beta}{v \cos \beta} = \frac{eu \sin \alpha}{u \cos \alpha},$$

or

$$\tan \beta = e \tan \alpha.$$

Hence both v and β are found.

Ex. A glass marble is projected against a horizontal pavement with a velocity of 20 *f/s*, and at an angle to the pavement whose sine is $\frac{3}{5}$: find the velocity and direction of the rebound, the coefficient of restitution being $\frac{1}{2}$.

Here, since the horizontal component of the velocity is unaltered, we have (the notation being the same as above)

$$v \cos \beta = u \cos \alpha = 20 \times \tfrac{4}{5} = 16,$$

and, since the vertical component of the velocity of recoil = e × vertical component of that of approach, we have

$$v \sin \beta = eu \sin \alpha = \tfrac{1}{2}\, 20 \times \tfrac{3}{5} = 6.$$

On squaring and adding the results, we have

$$v^2 (\sin^2\beta + \cos^2\beta) = 292, \quad \text{or} \quad v^2 = 292:$$

$$\therefore \quad v = 2\sqrt{73}.$$

Also, dividing the second equation by the first, we have

$$\tan \beta = \tfrac{6}{16} = \tfrac{3}{8},$$

$$\therefore \quad \beta = \tan^{-1} \tfrac{3}{8}.$$

Hence both v and β have been found.

291. *A smooth ball is projected at an angle α to a smooth horizontal plane: to find the connexion between the parabolic arcs described in successive rebounds.*

Let the body be projected with velocity u at an angle α to the horizon; then the horizontal component of its velocity $u \cos \alpha$ will remain constant during the whole period of its motion.

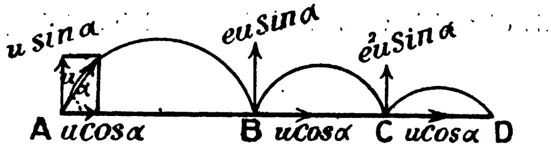


FIG. 119.

Since the vertical component of the velocity at A is $u \sin \alpha$, therefore at B it returns to the ground with the same vertical velocity, viz., $u \sin \alpha$; hence at B it rebounds with a velocity whose vertical component is $eu \sin a$, and of course the horizontal component is $u \cos a$. At C the vertical component of the rebound is $e^2 u \sin a$, and the horizontal component is still $u \cos a$. From these results it follows, as in Art. 243, that

the time of flight for the second parabolic arc $= \dfrac{2eu \sin a}{g}$,

the greatest height for the second parabolic arc $= \dfrac{e^2 u^2 \sin^2 a}{2g}$,

the range for the second parabolic arc $= \dfrac{eu^2 \sin 2a}{g}$,

and similarly for the other arcs; from which we conclude that if T, h, and R represent the time of flight, greatest height, and range respectively for the first arc, the values of these quantities for successive arcs are

times of flight $= T, eT, e^2 T$, &c. . . .
greatest heights $= h, e^2 h, e^4 h$, &c. . . .
ranges, $= R, eR, e^2 R$, &c. . . .

It is also evident that all the arcs have the same latus rectum, viz., $\dfrac{2u^2 \cos^2 a}{g}$ (Art. 249).

292. From these results it follows that after describing n parabolic arcs the total horizontal distance from the point of projection is

$$\frac{u^2 \sin 2a}{g}\left(1 + e + e^2 + \ldots e^{n-1}\right) = \frac{u^2 \sin 2a}{g} \cdot \frac{1 - e^n}{1 - e},$$

the limit of which when n is increased indefinitely, since $e^n = 0$, being

$$\frac{u^2 \sin 2a}{g(1 - e)} = \frac{R}{(1 - e)}.$$

From which we conclude that when the body reaches this distance from the point of projection it will cease to rebound, and will then slide along the plane with the constant horizontal velocity of $u \cos a$.

Ex. A smooth ball is projected with a velocity of 200 *f/s* at $\sin^{-1}\frac{4}{5}$ to a smooth horizontal plane: find the times of flight, greatest heights, and horizontal distances traversed for each of the parabolic arcs, given $e = \frac{1}{2}$. Also find at what point it will cease to rebound.

Here the initial vertical velocity $= u \sin \alpha = 200 \times \frac{4}{5} = 160$ *f/s*,
and the constant horizontal velocity $= u \cos \alpha = 200 \times \frac{3}{5} = 120$ *f/s*.
Hence, as in Art. 243, we find

time of flight $T = 10$ secs.,
greatest height $h = 400$ feet,
horizontal range $R = 1200$ feet.

Hence the following summary :—

Times of flight $= 10, 5, 2\frac{1}{2}$ . . . &c., seconds.
Greatest heights $= 400, 100, 25$ . . . &c., feet.
Horizontal ranges $= 1200, 600, 300$ . . . &c., feet.

Also it will cease to rebound and commence to slide when the horizontal distance from the point of projection is

$$\frac{u^2 \sin 2\alpha}{g(1-e)} \text{ or } \frac{R}{1-\frac{1}{2}} \text{ or } \frac{1200}{1-\frac{1}{2}} = 2400 \text{ feet.}$$

Oblique impact of two Smooth Spheres.

293. Let MN be the line joining the centres of the two spheres A and B at the moment of impact.

Let u and u' be the velocities of the two spheres before impact, and let the directions of u and u' make angles α and α' respectively with AB, the respective masses of the spheres being m and m'; also let v and v' be the velocities after collision, making angles β and β' respectively with AB. We are required to find v, v', β, and β' in terms of u, u', α, and α'.

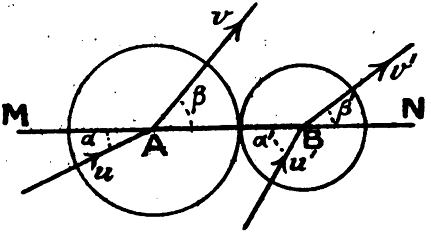


FIG. 120.

Now, since the spheres are smooth, the reaction between m and m' during impact, being perpendicular to their common surface at C, must be along the line AB. This reaction therefore will not alter the resolved parts of the velocities perpendicular to AB, these resolved parts being therefore constant. It is only the resolved parts of the velocities along AB that are altered by the impact, these being altered according to the same laws as in direct impact. Hence we have the following principles, which, being borne in mind, enable us to solve any problem involving oblique collision:—

(1) *The resolved part of each velocity perpendicular to the line joining the centres of the spheres after impact is equal to the resolved part in the same direction before impact.*

(2) *The algebraic sum of the momenta in the direction of the line joining the centres of the spheres after impact = the algebraic sum of the momenta in the same direction before impact.*

(3) *The velocity of recoil along AB = e × the velocity of approach along AB.*

Hence we have the following four equations:—

$$v \sin\beta = u \sin\alpha, \qquad (1)$$

$$v' \sin\beta' = u' \sin\alpha', \qquad (2)$$

$$mv \cos\beta + m'v' \cos\beta' = mu \cos\alpha + m'u' \cos\alpha', \qquad (3)$$

$$v \cos\beta - v' \cos\beta' = -e(u \cos\alpha - u' \cos\alpha'). \qquad (4)$$

These four equations will enable us in every case to determine the four quantities v, v', β, and β'.

Thus, multiply (4) by m' and add to (3), when we obtain

$$v \cos\beta = \frac{mu \cos\alpha + m'u' \cos\alpha' - m'e(u \cos\alpha - u' \cos\alpha')}{m + m'}.$$

Square this result, and add it to the square of (1), and, remembering that $\sin^2\beta + \cos^2\beta = 1$, we obtain

$$v^2 = u^2 \sin^2\alpha + \left\{\frac{mu \cos\alpha + m'u' \cos\alpha' - m'e(u \cos\alpha - u' \cos\alpha')}{m + m'}\right\}^2;$$

therefore v is found.

Also $\dfrac{v\cos\beta}{v\sin\beta}$ or $\cot\beta$

$$= \frac{mu\cos\alpha + m'u'\cos\alpha' - m'e\,(u\cos\alpha - u'\cos\alpha')}{(m+m')\,u\sin\alpha},$$

hence β is found.

Similarly, by eliminating v from equations (3) and (4), we can obtain the value of $v'\cos\beta'$, and then, with the aid of (2), we obtain the value of v' and $\cot\beta'$ as before.

Ex. A smooth sphere of mass 10 lbs. moving with a velocity of 16 *f/s* impinges on a smooth sphere of mass 6 lbs. having a velocity of 8 *f/s* in a direction at right angles to that of the first sphere. If, at the moment of impact, the line joining their centres is in the direction of the motion of the first sphere, find the magnitudes and directions of the subsequent velocities, the coefficient of restitution being $\frac{1}{3}$.

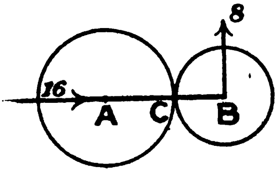


FIG. 121.

Here, since the velocity of the sphere A perpendicular to AB before impact $= 0$, therefore the velocity perpendicular to AB after impact $= 0$; hence the direction of motion of the sphere A is unchanged.

Again, the velocity of B perpendicular to AB before impact being $= 8$, hence the velocity of B perpendicular to AB after impact will be 8.

Now, since the sum of momenta along AB is the same before and after impact, taking v and v' to represent the subsequent velocities in the direction AB, we have

$$10v + 6v' = 10 \times 16 + 6 \times 0 = 160:$$

also

$$v - v' = -\tfrac{1}{3}(16 - 0),$$

or

$$3v - 3v' = -16.$$

Solving for v and v', we find $v = 8$ and $v' = 13\tfrac{1}{3}$. Hence the sphere A has a velocity of 8 *f/s* in the original direction of its motion, while B has a velocity of 8 *f/s* perpendicular to AB, and a velocity of $13\tfrac{1}{3}$ *f/s* along AB; the actual velocity of B is

$$\therefore\ = \sqrt{8^2 + 13\tfrac{1}{3}^2} = \frac{8\sqrt{34}}{3},$$

and whose direction is $\tan^{-1}\dfrac{3}{\sqrt{34}}$ with AB.

EXAMPLES XXXV.

1. A lump of putty of mass 4 ozs., and moving in a horizontal direction with a velocity of 16 feet per second, strikes against a mass of 2 lbs. resting on a smooth horizontal plane. Calculate the velocity with which they will move, the two masses being supposed to stick together after impact.

2. A sphere of mass 2 lbs., and which is moving with a velocity of 16 feet per second, impinges against another sphere of mass 1 lb. which is moving in the same direction with a velocity of 4 feet per second, and both bodies move with a common velocity after impact: find their common velocity and the kinetic energy lost.

3. If, in example 2, both bodies before impact are moving in opposite directions, find what their common velocity will be and the energy lost by the impact.

4. A mass of 20 lbs. moving with a velocity of 28 feet per second overtakes a mass of 6 lbs. moving with a velocity of 8 feet per second. If the coefficient of restitution be $\frac{1}{3}$, find the velocity of each after impact.

5. A sphere strikes a plane surface perpendicularly with a velocity of 60 feet per second: find the velocities of recoil when the coefficients of restitution between the sphere and the plane are respectively $\frac{1}{3}$, 0, and 1.

6. A sphere, let fall on a horizontal plane from a height of 18 feet, rebounds to a height of 2 feet: find the coefficient of restitution. Also find the height of the next rebound.

7. A sphere is let fall from a height h on a horizontal plane; if the coefficient of restitution be e, prove that the height of the rebound is e^2h.

8. An elastic ball of mass 10 lbs. is let fall from a height of 25 feet, and rebounds from a floor to a height of 16 feet: find the coefficient of restitution, and the average force exerted by the floor on the ball, if the time they are in contact is taken to be $\frac{1}{20}$ of a second.

9. With what velocity must a sphere A of mass 20 lbs. be projected against a sphere B of mass 40 lbs. moving in a direction opposite to that of A with a velocity of $18 f/s$, so that, taking the coefficient of restitution to be $\frac{1}{2}$, A's forward velocity will be changed to an equal backward one?

10. A sphere of mass 40 lbs. moving with a velocity of $17 f/s$ collides against one of mass 20 lbs. moving in the opposite direction with a velocity of $13 f/s$: taking the coefficient of restitution to be $\frac{1}{3}$, find the velocity of each after collision and the energy lost.

11. With what velocity must a ball of mass 6 lbs. be projected against one of 10 lbs. moving with a velocity of $20 f/s$, so that, taking the coefficient of restitution to be $\frac{1}{2}$, their velocities after impact may be equal?

12. A sphere of mass 28 lbs. having a velocity of 24 *f/s* impinges directly on a sphere of mass 14 lbs. with a velocity of $-$ 48 *f/s*; taking the coefficient of restitution to be $\frac{1}{3}$, find (*a*) the velocity of each after impact, (*b*) the total kinetic energy lost, (*c*) the total impulse, (*d*) the impulses during the periods of compression and restitution respectively, (*e*) the loss of kinetic energy during compression and the gain during restitution.

13. A sphere of mass 10 lbs. moving with a velocity of 32 *f/s* overtakes a sphere of mass 6 lbs. moving in the same direction with a velocity of 16 *f/s*; taking the coefficient of restitution to be $\frac{1}{3}$, find (*a*) the velocity of each after impact, (*b*) the total kinetic energy lost, (*c*) the total impulse, (*d*) the impulses during compression and restitution respectively, (*e*) the loss of kinetic energy during compression, and the gain during restitution.

14. Three spheres of masses m_1, m_2, and m_3, formed of the same substance, are placed with their centres in the same straight line; the first impinges directly on the second at rest, and then the second impinges on the third at rest: determine the coefficient of restitution in order that the velocity communicated to the third may be the same as if the first impinged directly on it, and with the same velocity as it impinged on the second.

15. A body let drop on to a horizontal plane continues to rebound until it finally comes to rest: prove that—

(*a*) The heights to which it rises in two consecutive rebounds are in the ratio $1 : e^2$.

(*b*) The velocities at the commencements to two consecutive rebounds are in the ratio $1 : e$.

(*c*) The times occupied in two consecutive rebounds are in the ratio $1 : e$.

16. A ball, let fall from a height of 16 feet, impinges on a horizontal plane and repeatedly rebounds: find when the ball will come to rest.

The body evidently takes 1 second to fall the 16 feet, and also strikes the horizontal plane with a velocity of 32 *f/s*. It then rebounds with a velocity $32e$; similarly the velocity of the second rebound is $32e^2$, &c.

The time of ascent in the first rebound is e, and therefore the total time up and down is $2e$; similarly the total time for second rebound is $2e^2$, &c. Hence the sum of the intervals occupied by the rebounds is

$$2e + 2e^2 + 2e^3 + \ldots \text{ seconds}$$
$$= 2e(1 + e + e^2 + \ldots),$$

which in the limit

$$= 2e\left(\frac{1}{1-e}\right) = \frac{2e}{1-e} \text{ seconds.}$$

Hence the total time from the moment it is let drop is

$$1 + \frac{2e}{1-e} = \frac{1+e}{1-e} \text{ seconds.}$$

17. A marble let drop from a height of 25 feet impinges on a horizontal pavement, the coefficient of restitution being $\frac{1}{2}$: find when the marble will come to rest.

18. A perfectly flexible chain is suspended by one end from a point with its lower end just touching a horizontal plane. If it be allowed to drop, prove that the pressure on the plane at any time during its fall is three times the weight of the portion coiled up on the plane.

19. Three spheres of masses m_1, m_2, and m_3 are situated with their centres in the same straight line, m_1 and m_2 being formed of the same substance. The mass m_1 is projected directly against m_2 with a given velocity, and then m_2 impinges on m_3; supposing the masses m_1 and m_3 to be given, find what should be the mass of m_2 in order that the velocity imparted to m_3 may be the greatest possible.

20. A sphere impinges directly on a sphere of equal mass at rest; show that the kinetic energy before impact is always less than twice the kinetic energy of the system after impact, unless the bodies are perfectly inelastic, when the kinetic energy before is equal to twice the kinetic energy after.

21. A ball falls from a height h on a horizontal plane and continues to rebound until it comes to rest: prove that the sum of the spaces described is equal to $h + \dfrac{2e^2h}{1 - e^2}$.

22. A smooth sphere is projected against a horizontal pavement with a velocity of $52 f/s$, and at an angle to the pavement whose sine is $\frac{5}{13}$. Find the velocity and direction of the rebound, the coefficient of restitution being $\frac{1}{2}$.

23. A smooth ball is projected with a velocity of $208 f/s$ at $\sin^{-1} \frac{5}{13}$ to a horizontal plane; find the times of flights, greatest heights, and horizontal distances traversed for each of the parabolic arcs, given $e = \frac{1}{2}$. Also find at what point it will cease to rebound.

24. A smooth sphere of mass 5 lbs., having a velocity of $20 f/s$, impinges on another of mass 10 lbs., moving in a direction at right angles to its own with a velocity of $10 f/s$. At the moment of impact the centre of the first is on the line of motion of the centre of the second. Determine the subsequent motion, if $e = \frac{1}{2}$.

CHAPTER XIX.

SIMPLE HARMONIC MOTION: THE PENDULUM.

294. If we imagine a point P to move along the circumference of a circle whose centre is O (fig. 122) with a uniform velocity of u feet per second, then, if AB be any diameter of the circle, and from P PM be drawn perpendicular to AB, then as P, starting from A, travels round the circle with uniform velocity, its projection M on the diameter AB will move to and fro along AB, from A to B and then from B to A, and will return to A in the same time as P takes to make a complete revolution of the circle.

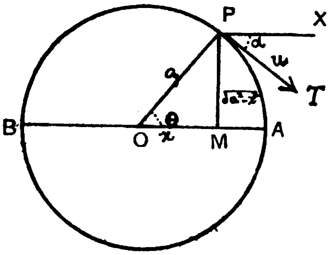


FIG. 122.

The motion of M is called a *Simple Harmonic Motion.*

A knowledge of the principles of Harmonic Motion is of very great importance to the student of physics, as the vibrations of light, heat, and sound—in fact, almost all small oscillations which occur in nature—are harmonic.

295. The **Amplitude** of a simple harmonic motion is the range on one side or the other of the middle point of the movement. Thus, OA or OB is the amplitude, and is therefore equal to the radius of the circle described by P.

A complete movement of M to and fro is called an **Oscillation.**

The **Period** is the time that elapses from the moment the moving point (M) is in any position until it is again in the same position and moving in the same direction. Hence the period is equal to the time taken to make a complete oscillation.

296. *To prove that the acceleration of a particle having a Simple Harmonic Motion about a fixed point varies as the distance of the particle from the fixed point, and is always directed towards that point.*

For the acceleration of P (fig. 122), when describing the circle with uniform velocity, is given by the equation

$$f = \omega^2 a \quad \text{(Cor. Art. 235)},$$

where a is the radius of the circle, and ω the angular velocity.

This acceleration is towards the centre O along the line PO.

Hence the acceleration of M is along the line MO, and is equal to the resolute of the acceleration of P along the line MO. Hence the acceleration of M

$$= \omega^2 a \cos\theta = \omega^2 a \frac{x}{a} = \omega^2 x.$$

Hence the acceleration of M varies as x, the distance of M from O.

297. *To show that the velocity* (v) *of the point M* (fig. 122) *at a distance x from O is given by the equation*

$$v^2 = \omega^2 (a^2 - x^2).$$

For the direction of the velocity u of P is along the tangent PT. Hence the velocity of M is the resolute of this velocity along a direction PX parallel to OM; that is, the velocity of M

$$= u\cos\alpha = u\sin\theta, \quad \text{for} \quad \alpha = 90^\circ - \theta;$$

but
$$\sin\theta = \frac{PM}{OP} = \frac{\sqrt{a^2 - x^2}}{a};$$

hence $$v = \frac{u\sqrt{a^2 - x^2}}{a},$$

or $$v^2 = \frac{u^2}{a^2}(a^2 - x^2);$$

but $$u = \frac{2\pi a}{T}; \quad \therefore \quad \frac{u^2}{a^2} = \left(\frac{2\pi}{T}\right)^2 = \omega^2;$$

hence $$v^2 = \omega^2(a^2 - x^2),$$

where a is the amplitude, and x the distance of the point M from the point O.

N.B.—It is evident from the above that, when the point M is at O, or, in other words, when $x = 0$, the velocity is greatest and the acceleration is zero; and when the point M is at A or B, i.e. when $x = a$, the velocity is zero, and the acceleration reaches its greatest value.

Ex. 1. The mass of an oscillating body is 16 lbs., the amplitude 4 feet, and the time of a complete oscillation $\frac{2\pi}{3}$: find in lbs. wt. the force acting on the mass when it is greatest, and also when the mass is 1 foot from the centre.

Here the angular velocity

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{2\pi}{3}} = 3.$$

Also, $$f = \omega^2 x, \text{ and } f \text{ is greatest when } x = a;$$

$$\therefore \text{ greatest value of } f = \omega^2 a = 3^2 \times 4 = 36 f/s/s.$$

Hence the greatest value of the force $P = 16 \times 36$ poundals $= 18$ lbs. wt.

Also, to find the value of P when the mass is 1 foot from the centre.

Here $$f = \omega^2 x = 3^2 \times 1 = 9,$$

and $$P = mf = 16 \times 9 \text{ poundals} = 4\tfrac{1}{2} \text{ lbs. wt.}$$

Ex. 2. If the velocity of a point is connected with its distance x from a fixed point by the relation

$$v^2 = 16\pi^2(169 - x^2),$$

find the amplitude and the time of an oscillation, feet and seconds being taken as units of space and time.

Here the amplitude a is the value of x when $v = 0$.

$$\therefore \quad 0 = 16\pi^2(169 - a^2);$$

$$\therefore \quad 169 - a^2 = 0, \quad \text{or} \quad a = 13.$$

Also, $$\omega^2 = 16\pi^2, \quad \therefore \quad \omega = 4\pi;$$

but the period $$T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \tfrac{1}{2} \text{ a second.}$$

298. *To find the time of an oscillation of a particle which has a Simple Harmonic Motion where the acceleration (f) of the particle is given by the equation $f = \mu x$, where x is the distance of the particle from a fixed point.*

Here $$f = \mu x,$$

but $$f = \omega^2 x \text{ (Art. 296)};$$

$$\therefore \quad \omega^2 = \mu, \quad \text{or} \quad \omega = \sqrt{\mu}; \quad \text{but} \quad \omega = \frac{2\pi}{T};$$

$$\therefore \quad \frac{2\pi}{T} = \sqrt{\mu}, \quad \text{or} \quad T = \frac{2\pi}{\sqrt{\mu}},$$

where $$\mu = \frac{f}{x} = \frac{\text{acceleration}}{\text{displacement}},$$

and where T is the time of an oscillation.

Note.—An oscillation is a *double vibration*; for an oscillation represents the whole movement to and fro, while a *vibration* is a movement in one direction only. Hence the period of a vibration is half that of an oscillation; that is, the period of a vibration

$$= \frac{\pi}{\sqrt{\mu}}.$$

Since the period for an oscillation is $\frac{2\pi}{\sqrt{\mu}}$, it follows that as long as the acceleration μx at a distance x from O is unchanged, or, in other words, as long as the force at a distance x from O is $m\mu x$, the *period of the oscillation is always the same and is independent of the amplitude.*

The oscillations of the particle are under these circumstances said to be **Isochronous.**

The Pendulum.

299. **Definition.**—A simple pendulum is a heavy particle suspended by a light inextensible thread from a fixed point.

300. *To prove that the period of a small oscillation of a simple pendulum is given by the equation*

$$T = 2\pi \sqrt{\frac{l}{g}},$$

where l is the length of the pendulum.

Note.—An oscillation is said to be *small* when the measure of the amplitude is so small that its square approaches so nearly to zero that it may be neglected. If O be the point to which the thread is attached, we may regard the motion of the bob at A as that of a particle sliding to and fro along a smooth circular arc AMA' (OM being the vertical drawn through O).

Now the weight mg of the particle can be resolved into—$mg \cos \alpha$ along AB the production of the radius, and $mg \sin \alpha$ along the tangent AC.

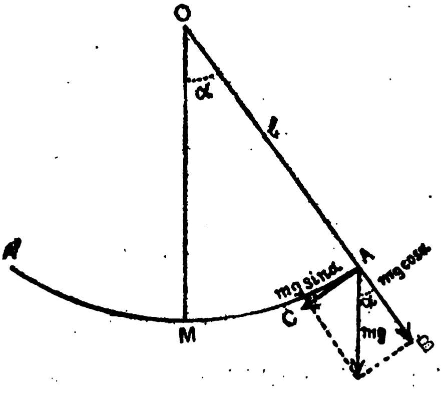


Fig. 123.

But $mg \cos \alpha$ is neutralized by the tension of the string. Hence, $mg \sin \alpha$ is the only force causing an acceleration. Hence, if f be the acceleration along the tangent, we have

$$mg \sin \alpha = mf \quad \text{or} \quad f = g \sin \alpha;$$

but since α is a very small angle, $\sin \alpha = \alpha$ (expressed in circular measure); $\therefore\ f = g\alpha$;

but $a = \frac{s}{l}$, where s = length of the arc AM,

$$\therefore\ f = \left(\frac{g}{l}\right) s = \mu s, \quad \text{where } \frac{g}{l} = \mu.$$

Hence, the period of an oscillation is given by the equation

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}},$$

and the time for a single vibration is therefore $\pi\sqrt{\frac{l}{g}}$.

301. Since the value we have obtained for T does not depend on the length of the arc AA', it follows that, provided the amplitude AM is *small*, the oscillations of a simple pendulum are isochronous.

302. The value of g at any place can be found by means of the pendulum. The time of making, say, 200 oscillations is noted; and this time being divided by 200 gives the period T for one oscillation. Then, if l be the length of the pendulum, we have

$$T = 2\pi\sqrt{\frac{l}{g}}, \quad \text{or} \quad T^2 = \frac{4\pi^2 l}{g}, \quad \text{or} \quad g = \frac{4\pi^2 l}{T^2},$$

from which the value of g can be calculated.

Note.—A pendulum which vibrates once per second is called *a seconds pendulum*.

Ex. Find the value of g in London [illegible]re the seconds pendulum is 39·1393 inches long;

here, $T = \pi\sqrt{\frac{l}{g}}$, and $T = 1$ second; $\therefore\ g = \frac{\pi^2 l}{T^2} = \pi^2 l$;

$$\therefore\ g = (3\cdot14159)^2 \times \frac{39\cdot1393}{12} = 32\cdot19 f/s/s.$$

303. **The Conical Pendulum.**—If the bob A of a pendulum, which is attached by means of a light flexible string of length l to a fixed point O, be made to revolve in a horizontal circle, the arrangement is called a *conical pendulum.*

Supposing the bob whose mass is m be made to describe a horizontal circle of radius r about a vertical axis OC with uniform velocity v. There must therefore be some force acting inwards towards the centre C which compels it to describe the circle. This force which

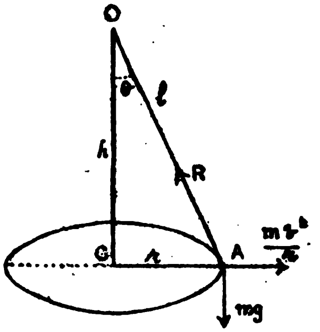


FIG. 124.

$$= \frac{mv^2}{r}$$

is the resultant of the tension of the string R, and the weight of the body mg.

Hence, if we imagine a force

$$= \frac{mv^2}{r}$$

applied outwards, [illegible]e ma[illegible]nsider the bob in equilibrium under the three forces :—

Its own weight mg [illegible]ng vertically downwards,

R the tension of the string,

and $\frac{mv^2}{r}$ acting horizontally outwards.

Resolving these forces vertically and horizontally we have

$$R \cos \theta = mg, \quad \text{and} \quad R \sin \theta = \frac{mv^2}{r} = \frac{mv^2}{l \sin \theta}.$$

Dividing one result by the other we have

$$\frac{\cos\theta}{\sin\theta} = \frac{gl\sin\theta}{v^2}, \quad \text{or} \quad v^2 = \frac{gl\sin^2\theta}{\cos\theta};$$

but, if T be the time of revolution,

$$vT = 2\pi r = 2\pi l\sin\theta;$$

$$\therefore \quad v^2 = \frac{4\pi^2 l^2\sin^2\theta}{T^2}; \quad \therefore \quad \frac{gl\sin^2\theta}{\cos\theta} = \frac{4\pi^2 l^2\sin^2\theta}{T^2},$$

and hence

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}.$$

And hence we conclude, by comparing this result with that in Art. 300, that a conical pendulum revolves in the same time as that taken for an oscillation of a simple pendulum whose length is the vertical projection of the conical pendulum.

Ex. A particle of mass 4 lbs. is tied to a fixed point by a string 17 feet long, and moves in a horizontal circle of radius 15 feet: find its velocity, the tension of the string, and the time of revolution.

Here (fig. 124)

$$R\cos\theta = 4g = 128, \quad \text{or} \quad \frac{R8}{17} = 128; \quad \therefore \ R = 272 \text{ poundals} = 8\tfrac{1}{2} \text{ lbs. wt.};$$

also

$$\frac{mv^2}{r} = R\sin\theta, \quad \text{or} \quad \frac{4v^2}{15} = \frac{272\times 15}{17}; \quad \therefore \ v = 30 \text{ feet per second.}$$

Also

$$T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{8}{32}} = \pi \text{ secs.}$$

EXAMPLES [illegible]XVI.

1. Calculate the length of the seconds pendulum in Paris where

$$g = 32{\cdot}182 f/s/s.$$

2. Given that the length of a half-seconds pendulum at Spitzbergen is 9·8036 inches, find the acceleration of gravity at that place.

3. A heavy ball, suspended by a fine wire, vibrates in a small arc; 48 vibrations are counted in 3 minutes: find the length of the wire.

4. In a harmonic oscillation of a particle whose mass is 6 lbs., if the amplitude is 25 feet, and the time of a complete oscillation 6 seconds, find the velocity of the particle and the force acting on it when it is 7 feet from the centre of force.

5. A ball is moving in a horizontal circle, being suspended by a fine thread from a point vertically above the centre; show that the point must be at a height of

$$\frac{g}{4\pi^2 n^2}$$

above the plane of the circle in order that motion may go on steadily, where n is the number of revolutions made in one second.

6. The mass of an oscillating body is 10 lbs., the amplitude is 17 feet, and the time of oscillation $\frac{\pi}{3}$ seconds: find in lbs. wt. the force acting on the mass (1) when it is greatest, (2) when the mass is 8 feet from the centre. Also find the velocity when the mass is 8 feet from the centre.

7. If the velocity of a point is connected with its distance x from a fixed point by the relation

$$v^2 = 9\pi^2 (1681 - x^2),$$

find the amplitude and the time of an oscillation, feet and seconds being the units of space and time respectively.

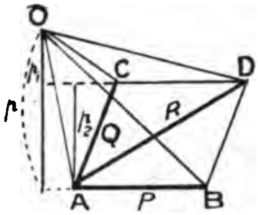
8. A particle of mass 10 lbs. is tied to a fixed point by a string 25 feet long, and moves in a horizontal circle of radius 24 feet: find its velocity, the tension of the string, and the time of one revolution.

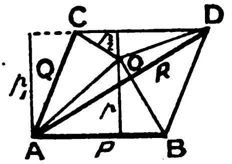
APPENDIX.

To prove that the algebraic sum of the moments of two concurrent forces about any point in their plane is equal to the moment of their resultant.

Alternative Proof.

Case I.—Let the point O lie outside the angle BAC formed by the lines of action of the forces P and Q. Join OA, OB, OC, and OD. Let p and p_1 be the lengths of the perpendiculars dropped from O on AB and CD respectively, and p_2 the length of the perpendicular drawn from A on CD.





Now

$$2 \triangle OAB = AB \times p; \text{ but } AB = CD, \text{ and } p = p_1 + p_2$$

$$= CD \times p_1 + CD \times p_2;$$

or $$2 \triangle OAB = 2 \triangle OCD + 2 \triangle ACD$$

(to each side add $2 \triangle OAC$);

$$\therefore \quad 2 \triangle OAB + 2 \triangle OAC = 2 \triangle OAD;$$

but $2 \triangle OAB$ = moment of P about O,

$2 \triangle OAC$ = moment of Q about O,

and $2 \triangle OAD$ = moment of R about O,

or moment of P + moment of Q = moment of R.

Case II.—Where O lies within the angle BAC. The construction is as before.

$$2 \triangle OAB = AB \times p; \text{ but } AB = CD \text{ and } p = p_1 - p_2$$

$$= CD \times p_1 - CD \times p_2,$$

or $$2 \triangle OAB = 2 \triangle ACD - 2 \triangle OCD$$

(from each side take $2 \triangle OAC$);

$$\therefore \quad 2 \triangle OAB - 2 \triangle OAC = 2 \triangle OAD,$$

or moment of P − moment of Q = moment of R.

Q. E. D.

ANSWERS

I. (Page 6.)

1. 224 inches. **2.** $\frac{Pb}{a}$. **3.** $\frac{Qa}{P}$ **4.** 264 lbs. wt.

II. (Pages 13, 14.)

1. 25 lbs. wt. **2.** 60 lbs. wt.

3. $2\sqrt{2}$ lbs. wt., in a direction south-west.

4. 8 lbs. wt. and 2 lbs. wt. respectively; 7 lbs. wt.

5. 10 lbs. wt., whose line of action bisects the angle between the forces.

6. 120°.

7. $2\sqrt{3}$ lbs. wt., in a direction perpendicular to that of the 2 lbs. force.

8. 15 and 25 lbs. wt. respectively. **9.** $10\sqrt{3}$ lbs. wt.

11. 2 lbs. wt. **12.** 45°.

13. 120° between $2\sqrt{3}$ and $\sqrt{3}$; 90° between $\sqrt{3}$ and 3; 150° between 3 and $2\sqrt{3}$. **14.** 16·05 lbs. wt.

15. (1) In the same direction as the resultant of the other forces; (2) in the opposite direction to that of the resultant of the other forces.

16. In a direction opposite to that of the force which has been removed.

III. (Pages 20, 21.)

2. 4 units parallel to the side whose length is 5 inches.

3. $\sqrt{3}$ units; 90°. **4.** Cor. 2, Art. 25.

6. 105°, 120°, and 135° opposite the forces $\sqrt{3}+1$, $\sqrt{6}$, and 2 respectively.

7. A force represented by CD; a force represented by DC.

8. Apply Art. 25. **9.** $\sqrt{3} : 1 : 1$.

10. $\frac{20\sqrt{3}}{3}$; tension $= \frac{40\sqrt{3}}{3}$. **11.** Apply Art. 25.

12. Apply Art 21. **16.** The intersection of the medians.

IV. (PAGE 30.)

1. Each component is = $10\sqrt{2}$ lbs. wt. 2. $40\sqrt{3}$ lbs. wt.

3. 240 lbs. wt. 4. Apply Arts. 32, 33.

5. 52 lbs. wt., whose direction is at $\tan^{-1}\frac{5}{12}$ to AB.

6. 28 lbs. wt., and 15 lbs. wt. respectively. 7. 11·16 lbs. wt.

8. 10 lbs. wt.; tension = 26 lbs. wt.

V. (PAGES 37, 38.)

1. 88 units of moment; 264 units of moment.

2. $92\frac{4}{13}$, 360, and 200 units of moment round A, B, and C respectively.

4. 30 lbs. wt. 5. 60 lbs. wt. 7. $166\frac{2}{3}$ lbs. wt.

VI. (PAGES 46, 47.)

1. 2 feet from the 4 lbs. wt. 2. $4\frac{1}{2}$ feet from the 3 lbs wt.

3. 11 lbs. 4. $1\frac{3}{4}$ lbs. 5. $3\frac{2}{5}$ and $4\frac{3}{5}$ tons wt.

6. 60 and 24 lbs. wt. respectively.

7. Resultant is an upward force of 4 lbs. wt., acting at a point 8 inches from the fixed point. The moments are 20 and -8 units respectively, the distances being measured to the right of the fixed point.

8. $2\frac{2}{11}$ feet from one end. 9. 11 feet and 15 feet from the same end.

10. 5 cwt. 11. 7 feet from the edge, i.e. just to the end of the plank.

13. $2\sqrt{5}$ lbs. wt. The direction makes an angle θ with CD, such that $\tan\theta = 2$. The distance of its line of action from $C = \frac{3a}{2\sqrt{5}}$, where a is the length of the side of the square.

14. 5 lbs. wt. The direction makes an angle θ with AB such that $\tan\theta = \frac{3}{4}$. The distance of its line of action from A is $3a$, where a is the length of the side of the square.

VII. (PAGES 53, 54.)

2. 6 lbs. wt. In the same direction as the force of 6 lbs. wt., at a perpendicular distance from it of 2 feet. 3. 36 ft.-lbs.

6. A force of 13 lbs. wt., whose line of action is distant 10 feet from the intersection of the axes, and whose direction is inclined to the axis of x at $\tan^{-1}\frac{12}{5}$. 7. The magnitude of the resultant is 10; its direction is inclined to the axis of x at $\tan^{-1}\frac{3}{4}$; and its line of action is distant 3·9 from the intersection of the axes.

8. No. The resultant is $3\sqrt{2}$, inclined at an angle 45° to the lines of action of 15 or 18, and its line of action is distant $26\sqrt{2}$ from the point of intersection of the 18 and 27 forces.

9. The force must be applied at a point in the bar at a distance from C measured along CA of $\frac{1}{5}l$, where l is the length of the bar. Its magnitude is 5 lbs. wt., its direction being upwards and parallel to the given force of 5 lbs. wt.

VIII. (Page 64.)

For **1**, **2**, and **3**, apply Arts. 68, 69. **4.** 3 : 1. **6.** $9\frac{3}{17}$ ft. from O.

7. $1\frac{5}{52}$ inches from the line of junction of the two pieces.

8. $7\frac{1}{3}$ feet from A. **9.** $1\frac{1}{5}$ inches from the centre of the larger circle.

10. $\frac{2}{3}$ of an inch from the centre of the larger sphere.

IX. (Pages 69, 70.)

1. At a point distant $\frac{1}{8}r$ from the centre of the larger plate, r being the radius of the larger plate.

2. Its centre must be 15″ from the centre of the disk.

3. At a distance $\frac{d}{12}$ from the centre, where d is the length of the diagonal which passes through the portion cut out.

4. At a distance $\frac{a}{9}$ from the centre, where a is the length of a side of the parallelogram.

5. $\frac{d}{21}$, where d is the length of the diagonal passing through the vertex of the triangle.

6. At a distance $\frac{l}{9}$ from the C. G. of the triangle, where l is the length of the median.

7. At a distance $\frac{l}{18}$ from the C. G. of the triangle, where l is the length of the median.

10. It is distant $\frac{a}{2}$ from AD, and $\frac{19a}{30}$ from AB, where a is the length of the side of the square.

11. $\frac{2\sqrt{7}}{7}a$, where a is the length of a side of the triangle.

12. It is distant $\frac{8}{9}$ of a foot from BC, and 1 foot from CA.

13. It is distant $1\frac{1}{28}$ inches from the left vertical wire, and 4 inches above the base.

14. Taking the sides in the order of their magnitudes, commencing with the greatest, the distances of the C. G. from them are $\frac{8}{13}$, 1, and 8 inches respectively.

15. At a height of 15 units of length above the base whose length is 17.

X. (Page 74.)

1. A weight equal to that of the table.

2. A weight equal to that of the table. **4.** 60°.

6. $\frac{\sqrt{10}}{10}$ feet.

XI. (Pages 82–84.)

1. 12 lbs. wt.; tension = 20 lbs. wt. **2.** 48 and 20 lbs. wt. respectively.

3. 80 and 18 lbs. wt. respectively. **4.** $\frac{Wl}{2\sqrt{l^2 - a^2}}$. **5.** $\frac{10\sqrt{3}}{3}$ oz. wt.

6. Tension = Reaction = 10 lbs. wt. **7.** Tension = Reaction = 17½ lbs. wt.

8. (1) $\frac{5\sqrt{3}}{3}$, (2) $\frac{\sqrt{35}}{7}$ feet. **9.** 20 lbs. wt. **10.** 26 lbs. wt.

11. Tension = $\frac{W}{2}$, where W is the weight of the ladder.

12. Because the three forces acting on the ladder could not meet in a point.

14. 60 lbs. wt. **15.** 120 lbs. wt. **16.** $\frac{W \sin \beta}{\sin (\alpha + \beta)}$ and $\frac{W \sin \alpha}{\sin (\alpha + \beta)}$.

17. (*a*) $\frac{W}{2}$, (*b*) $\frac{W}{2\sqrt{3}}$.

19. Tensions are 20 and 34 lbs. wt. respectively; the greatest horizontal force is 56 lbs. wt. **20.** 45°.

XII. (Page 91.)

1. $6\frac{6}{7}$ inches from the 3-lb. force. **2.** 2 lbs. **3.** 5½ feet. **4.** 7 : 9.

5. $P = 21$, and $W = 27$, supposing the lever to be worked at a mechanical advantage.

6. 1½ feet. **7.** 5 lbs. wt.

8. The fulcrum is distant $\frac{48\left(8 - 3\sqrt{2}\right)}{23}$ inches from the extremity at which the force of 3 units acts.

10. 33⅓ lbs. wt.

XIII. (Pages 97, 98.)

1. 7 lbs. wt.; 7 : 8. **2.** $4\sqrt{15}$ lbs. **3.** $\frac{5}{21}$, or ·24 per cent. nearly.

5. 14¼ lbs. wt. **6.** 8 lbs. wt. **7.** (1) 8*s*. $6\frac{2}{5}$*d*., (2) 7*s*. 6*d*.

8. 5⅓″ from the fulcrum; 4″. **9.** 21⅓″ from the fulcrum.

10. 28 lbs.; $15\frac{3}{5}$ lbs. **11.** 3¾″ from the end.

XIV. (PAGES 102, 103.)

1. 3′ 6″. **2.** $13\frac{1}{2}$ lbs.; $16\frac{1}{2}$ lbs. wt.

3. The radius of the wheel must be 12 times that of the axle.

4 The radius of the wheel : radius of the axle : : 11 : 2. **5.** $2\frac{2}{3}''$; 9

6. $96\frac{4}{5}$ lbs. **7.** 24 cwt. **8.** 5 lbs. wt. **9.** $R = 4'$, $r = 6''$.

10. 56 lbs. wt. **11.** 10 lbs. wt. **12.** $12\frac{26}{27}$ feet.

13. 960 lbs. **14.** $R = 33''$ and $r = 3''$.

XV. (PAGES 108, 109.)

1. 50 lbs. wt.; 2. **2.** 54 lbs. wt. **3.** 18 lbs. wt. **4.** 5 lbs. wt.

5. 35 lbs. wt. **6.** 32 lbs. wt.; 30 lbs. wt. **7.** Three movable pulleys.

8. 7. **9.** 3 movable pulleys; (*a*) 6 lbs. wt., (*b*) $6\frac{2}{3}$ feet.

10. Four movable pulleys; 8 lbs. wt. **11.** $4\frac{1}{2}$ ozs. wt.; 3 inches.

12. $144\frac{1}{2}$ lbs. wt. **13.** $168\frac{3}{4}$ lbs. wt. **14.** 2 lbs. **15.** 3.

16. 6 stones wt. **17.** 4 stones wt. **18.** 21 lbs. wt.

XVI. (PAGE 110.)

1. 8; 4. **2.** 5. **3.** $n = 8$; $w = 15$. **4.** 4 stones wt., or 56 lbs. wt.

5. 52 lbs. wt. **6.** 3 stones wt. **7.** $\frac{W}{9}$.

XVII. (PAGES 113, 114.)

1. 2 movable pulleys; (*a*) 78 lbs.

2. 3 movable pulleys; (*a*) 183 lbs.; (*b*) $7\frac{1}{2}$ feet.

3. 8 lbs. wt. **4.** 10 lbs. wt. **5.** $58\frac{3}{4}$ lbs. **6.** 136 lbs. wt.

7. 32 lbs. wt.

8. $2\frac{1}{5}$ inches from the point at which the string whose tension is greatest is attached.

XVIII. (PAGE 115.)

1. 244 lbs.; 260 lbs. wt. **2.** 10 lbs. wt.; $\frac{4}{9}$ of a foot.

4. Two movable blocks of the 4th system, and one of the 1st system.

XIX. (PAGE 117.)

2. 20 lbs. wt.; 48 lbs. wt. **3.** 18 lbs. wt. **4.** 30°; $\frac{W\sqrt{3}}{2}$.

5. 60 : 61.

XX. (PAGE 119.)

2. 45°; $W\sqrt{2}$. 3. 60°; $W : P :: 1 : \sqrt{3}$. **4.** 11 : 61; 122 lbs. wt.

XXI. (PAGES 120, 121.)

1. $P = 2$ lbs. wt.; $R = 42$ lbs. wt.

4. Each is equal to the weight of the bead.

6. $\dfrac{100\sqrt{6}}{3}$ lbs. wt.; $50\left(\sqrt{2} - \dfrac{\sqrt{6}}{3}\right)$ lbs. wt.

7. $7\frac{1}{2}$ lbs. wt.; $28\frac{1}{2}$ lbs. wt.

XXII. (PAGE 125.)

1. 9051·4 lbs. wt. **2.** 7200 lbs. wt. **3.** 2112. **4.** $3\frac{2}{11}$ lbs. wt.

5. 21 inches. **6.** 264 lbs. wt. **7.** $\frac{1}{3}$ of an inch, or 3 threads to the inch.

XXIII. (PAGES 133–135.)

1. 3 ozs. wt. **2.** 100·8 lbs. wt.; 846 lbs. wt. nearly.

4. $\dfrac{\sqrt{3}}{3}$. **5.** $\dfrac{60\sqrt{2}}{13}$ lbs. wt. **6.** (1) 32 lbs. wt.; (2) 8 lbs. wt.

7. (1) $10\left(2\sqrt{3} - 1\right)$ lbs. wt.; (2) $10\left(2\sqrt{3} + 1\right)$ lbs. wt.

9. (1) 169 lbs. wt.; (2) 119 lbs. wt. **10.** 5 ozs. wt.

12. 440 lbs. wt.; the resultant reaction acts at a point of the plane $1\frac{3}{4}$ ft. from the centre of the lower side of its upper face measured downwards along the line of greatest slope. **13.** $8\frac{1}{2}$ feet. **14.** 30 lbs. wt.

17. $\mu = \frac{2}{3}$; normal reaction of wall $= 23\frac{1}{13}$ lbs. wt.; normal reaction of ground $= 34\frac{8}{13}$ lbs. wt.

18. Inclination $= \tan^{-1}\frac{3}{5}$.

XXIV. (PAGES 147, 148.)

1. 14 ft. per sec.; 18 ft. per sec.; 250 ft. per sec.

2. (*a*) 6 ft. per sec.; (*b*) 0.

3. 4 ft. per sec.; 8 ft. per sec.; 240 ft. per sec.

4. 11 seconds. **5.** 22 seconds. **6.** 50 feet; 20 ft. per sec.

7. 4 *f/s/s*. **8.** 11 ft. per sec. **9.** 16 feet.

10. A retardation of 40 *f/s/s*. **11.** $3\frac{1}{8}$ *f/s/s*.

12. (*a*) A retardation of 8 *f/s/s*; (*b*) 121 feet. **13.** 4400 feet.

14. A retardation of 32 *f/s/s*. **15.** 20 seconds; 600 metres.

16. 136 feet. **17.** 13 feet. **18.** $u = 10$ feet per second; $f = 6$ *f/s/s*.

19. 6 seconds. **20.** No.

XXV. (PAGES 152–154.)

1. (1) 5 seconds; (2) 10 seconds; (3) 400 feet.

2. 44 feet per second, or 30 miles per hour.

3. (1) 1296 feet; (2) 272 feet. **4.** 2 or 3 seconds.

5. 88 feet per second, or 60 miles per hour.

7. 2 seconds, or $4\frac{1}{4}$ seconds; 136 feet. **8.** 96 feet.

9. 100 feet per second. **10.** 2 seconds. **11.** 84 feet.

12. $1\frac{1}{3}$ seconds from the time the first stone is let drop.

13. $124\frac{4}{5}$ feet per second.

14. In 3 seconds after the instant at which the first stone is projected; 24 feet below the point of projection.

15. In $1\frac{1}{4}$ seconds; 25 feet from the top. **17.** 32·2 *f/s/s*.

18. The balloon was descending with a velocity of 20 feet per second.

19. 516 feet. **20.** 20 feet per second; 1400 feet. **21.** 4 feet.

22. 981 cms. per second per second. **23.** 490·5 cms.; 2 seconds.

24. 1120 feet per second. **25.** In 2 seconds.

27. $\frac{5}{18}$ ft. **28.** 784 feet.

XXVI. (PAGES 159–161.)

1. (*a*) 8 *f/s/s*; (*b*) 11 seconds; (*c*) 484 feet. **2.** $1\frac{3}{4}$ feet.

3. (*a*) 864 poundals; (*b*) 27 lbs. wt. **4.** $1\frac{1}{2}$ lbs. wt.

5. $25\frac{3}{5}$ lbs. **6.** 1 ton. **7.** 40 seconds.

8. 5 seconds; 800. **9.** 5 tons wt. **10.** $2\frac{1}{2}$ lbs. wt.

11. $3\frac{1}{3}$ tons wt. **12.** 1694 feet.

13. (*a*) $2\frac{1}{2}$ lbs. wt.; (*b*) $22\frac{1}{2}$ lbs. wt. **14.** 140 lbs. wt.; 2380 lbs. wt.

15. Force : wt. of train : : 11 : 480. **16.** 32 feet.

17. 2 ozs. wt. **18.** 10 ozs. wt. **19.** 312 feet (144 + 96 + 72).

20. 17·658 metres. **21.** $1\frac{1}{16}$ lbs. wt.

22. (*a*) $\frac{1}{16}$ lb. wt., or 2 poundals; (*b*) $\frac{1}{32}$ lb. wt., or 1 poundal.

23. (*a*) 2 dynes; (*b*) 1 dyne.

XXVII. (PAGES 169–171.)

1. $8 f/s/s$. **2.** 30°. **3.** 40 feet per second. **4.** 50 feet; 5 seconds.

5. -2 feet, which means that the body slides *downwards* through 2 feet in the 12th second.

6. 80 feet per second; 80 feet per second; no (see Art. 200).

7. See Art. 200.

8. The divisions counting from above downwards are 2, 6, and 10 feet respectively.

10. 19,440 lbs. wt. **11.** 7480 lbs. wt.

12. $11\frac{1}{4}$ feet; it will slide down; $12\sqrt{2}$ feet per second.

13. (*a*) $4\frac{17}{80}$ tons wt.; (*b*) $\frac{17}{24}$ tons wt. **14.** (*a*) 50 feet; (*b*) $2\frac{1}{2}$ seconds.

15. $f = 8 f/s/s$, and $\mu = \frac{1}{4}$. **16.** 32,000 lbs. wt.

17. The direction of the line of quickest descent is got by joining the given point to the lowest extremity of the vertical diameter of the circle.

XXVIII. (PAGE 173.)

1. (*a*) 40 lbs. wt.; (*b*) 40 lbs. wt.; (*c*) 40 lbs. wt.; (*d*) 50 lbs. wt.; (*e*) 30 lbs. wt.; (*f*) 80 lbs. wt.; (*g*) zero.

2. (*a*) 126 lbs. wt.; (*b*) 98 lbs. wt.

3. (*a*) 98 lbs. wt.; (*b*) 126 lbs. wt. **4.** Zero.

5. (*a*) Vertically downwards, with an acceleration of g; (*b*) vertically upwards, with an acceleration of g.

6. (*a*) Either the lift is at rest, or moving upwards or downwards with a uniform velocity; (*b*) either the lift is moving upwards with a velocity which is increasing at the rate of 16 foot-second units, or downwards with a velocity which is decreasing at the same rate.

7. $2 f/s/s$ vertically downwards.

XXIX. (PAGES 184–186.)

1. $2 f/s/s$; $7\frac{31}{32}$ ozs. wt.; 5 feet. **2.** $7\frac{1}{5}$ ozs. wt.; 9 ozs. wt.

3. 3 : 1. **4.** $1\frac{31}{32}$ lbs. wt.; $1\frac{3}{4}$ lbs. wt. **5.** $6\frac{1}{4}$ feet; $\frac{5}{8}$ second.

6. 3 feet. **7.** 28 feet; $4\frac{1}{2}$ lbs. wt. and 3 lbs. wt. respectively.

8. (1) $12 f/s/s$; (2) $3\frac{3}{4}$ lbs. wt,; (3) 30 feet. **9.** $1\frac{1}{2}$ feet.

10. 4 feet.; 3 lbs. wt.; $2\frac{1}{4}$ lbs. wt.

11. $4 f/s/s$; $2\frac{5}{8}$ lbs. wt. and $\frac{7}{8}$ lb. wt. respectively.

12. 3 seconds; 12 feet per second. **13.** 4 feet per second; $5\frac{1}{4}$ lbs. wt.

14. $\mu = \frac{1}{14}$; $T = 1\frac{7}{8}$ lbs. wt. **15.** $16 f/s/s$; $\frac{m}{2}$.

16. $16 f/s/s$; $19\frac{1}{2}$ grammes wt.; 24 grammes wt.

17. 3 feet per second; $11\frac{181}{180}$ lbs. wt. **18.** $5\frac{5}{6}$ feet.

19. 8 feet per second. **20.** 18 feet.

21. $14\frac{1}{2}$ feet; both masses will now slide down, the 15 and 17 lbs. masses having accelerations of $\frac{2g}{5}$ and $\frac{g}{5}$ respectively.

22. 3 feet; $T = 1\frac{3}{4}$ lbs. wt., and $T' = 2\frac{1}{2}$ lbs. wt., where T is the tension of string between the first and second, and T' that of string between the second and third weight. **23.** $4\frac{1}{4}$ feet.

24. The other mass ascends at the same rate as the monkey; and, supposing them both at first to start from the same level, they both will reach the pulley at the same instant. **25.** 18 feet.

XXX. (Pages 193, 194.)

1. 13 feet per second.

2. (*a*) 30 miles per hour; (*b*) 0; (*c*) 15 miles per hour.

3. $1 : \sqrt{3}$. **4.** $\sqrt{2} : 1$.

5. He should swim in a direction at right angles to the direction of the current.

6. At an angle of 135° to the direction of the motion of the train; $30\sqrt{2}$ miles per hour.

7. $6\frac{1}{4}$ lbs. wt., whose direction is at an angle θ to the south of east such that $\tan\theta = \frac{3}{4}$.

8. 10 lbs. wt., acting at $\tan^{-1}\frac{4}{3}$ to the south of east.

9. 5 miles. **10.** $\frac{1}{5}$ of a mile.

11. 13 lbs. wt., at $\tan^{-1}\frac{12}{5}$ to the west of north.

XXXI. (Pages 202, 203.)

2. $5 f/s/s$; $\frac{5}{8}$ lb. wt. **3.** $6\frac{1}{4}$ tons wt. **4.** $2\frac{1}{2}$ lbs. wt.

5. 16 feet per sec. **6.** $21\frac{3}{32}$ lbs. wt.

7. $f = 4f/s/s$ towards the centre; tension $= \frac{1}{8}$ lb. wt. **8.** 2 seconds.

10. 1 : 3436 nearly. **12.** 24 ft. per sec.

13. $\frac{3}{4}$ of its mass must be removed. **14.** $3\frac{1}{8}$ lbs. wt.; 100 feet.

15. 8 feet. **16.** 1 foot vertical fall.

XXXII. (PAGES 216, 217)

1. 2500 feet. **2.** $2346\frac{2}{3}$ ft. per sec. **4.** 240 feet; $2\frac{1}{2}$ secs.; 25 feet.

5. At the foot of the mast; $v = \sqrt{22^2 + 80^2} = 83$ ft. per sec. nearly, and inclined to the vertical at $\tan^{-1} \frac{11}{40}$.

6. 80 ft. per sec. **7.** 45°.

8. In 3 seconds it will be at a vertical height of 96 feet, and at a horizontal distance of 144 feet from the point of projection; 100 feet; 240 feet; 5 seconds.

9. It moves parallel to itself.

11. It strikes the post at a height of 44 feet, and at $\tan^{-1} \frac{36}{7}$ to the vertical.

12. 48 ft. per sec. and 40 ft. per sec. respectively.

14. At a horizontal distance of 19680 feet from the point of projection; it strikes the plane at $\tan^{-1} - \frac{7}{5}$.

15. The horizontal and vertical components of its initial velocity are 64 *f/s* and 56 *f/s* respectively, *i.e.* its initial velocity is $8\sqrt{113}$ *f/s*, and the angle of elevation is $\tan^{-1} \frac{7}{8}$.

16. The horizontal and vertical components of its initial velocity are 48 *f/s* and 96 *f/s* respectively, *i.e.* the initial velocity is $48\sqrt{5}$ *f/s*, and the angle of elevation is $\tan^{-1} 2$.

17. The focus is distant from the point of projection 15000 feet measured horizontally, and at a vertical height above it of 10000 feet. The directrix is a horizontal line 15625 feet above the point of projection.

23. 15°. **24.** $u \cos \alpha \sec \beta$. **25.** $\tan^{-1} (2 \tan \beta + \cot \beta)$; no.

XXXIII. (PAGES 225, 226.)

1. 2240 ft.-lbs. **2.** 5400000 ft.-lbs. **3.** $27\frac{1}{2}$ ft. per sec.

4. 100800000 ft.-lbs. **5.** 520 ft.-lbs. **6.** 520 ft.-lbs. **7.** $1\frac{79}{96}$ hours.

8. 179·2. **9.** 261072000 ft.-lbs. **10.** 202·4.

XXXIV. (PAGE 230.)

1. 44800; $12\frac{1}{2}$ tons wt. **2** $4\frac{13}{28}$ ft. per sec. **3.** $2\frac{6}{7}$ ft. per sec.; $\frac{25}{49}$ ft.

4. (1) 54 and 96 ft.-lbs.; (2) 96 and 54 ft.-lbs.; (3) 150 and 0 ft.-lbs.

5. (1) 2560 and 1440 ft.-lbs.; (2) 0 and 4000 ft.-lbs.

XXXV. (PAGES 246–8.)

1. $1\frac{7}{8}$ ft. per sec. **2.** 12 ft. per sec.; $1\frac{1}{2}$ ft.-lbs.

3. $9\frac{1}{3}$ ft. per sec.; $4\frac{1}{6}$ ft.-lbs. **4.** $21\frac{1}{13}$ and $28\frac{20}{39}$ ft. per sec. respectively.

5. 20, 0, and 60 ft. per sec. respectively. **6.** $\frac{1}{3}$; $\frac{2}{9}$ ft.

8. $\frac{4}{5}$; 460 lbs. wt. **9.** 18 ft. per sec.

10. 5 and 11 ft. per sec. respectively; 180 ft.-lbs. **11.** 44 ft. per sec.

12. (*a*) $-8\,f/s$ and $16\,f/s$ respectively; (*b*) 672 ft.-lbs.; (*c*) 896; (*d*) 672 and 224 respectively; (*e*) 756 ft.-lbs. loss and 84 ft.-lbs. gain.

13. (*a*) $23\,f/s$ and $31\,f/s$ respectively; (*b*) $11\frac{1}{4}$ ft.-lbs.; (*c*) 90; (*d*) 90 and 30 respectively; (*e*) 15 ft.-lbs. loss and $3\frac{3}{4}$ ft.-lbs. gain.

14. $\dfrac{m_2^2 + m_1 m_3}{m_2(m_1 + m_3)}$. **17.** In $3\frac{3}{4}$ secs. **19.** $\sqrt{m_1 m_3}$.

22. $2\sqrt{601}\,f/s$; the direction makes an angle $\tan^{-1}\frac{5}{24}$ with the plane.

23. Times of flight are 5, $2\frac{1}{2}$, $1\frac{1}{4}$, . . . etc., secs.
Greatest heights are 100, 25, $6\frac{1}{4}$, . . . etc., feet.
Horizontal ranges are 960, 480, 240, . . . etc., feet.

24. The 5-lb. mass after impact has a velocity of $10\sqrt{5}\,f/s$ in a direction inclined at $\tan^{-1}\frac{1}{2}$ to its original direction, while the 10 lb. mass has a velocity of $5\,f/s$ in the same direction as its original line of motion.

XXXVI. (PAGES 256, 257.)

1. 39·128 inches. **2.** $32{\cdot}2566\,f/s/s$. **3.** 45·87 feet.

4 $8\pi\,f/s$; $\dfrac{7\pi^2}{48}$ lbs. wt. **6.** $191\frac{1}{4}$ lbs. wt.; 90 lbs. wt.; $90\,f/s$.

7. 41 feet; $\frac{2}{3}$ of a second.

8. $v = 96\sqrt{\frac{2}{7}}\,f/s$; tension $= 35\frac{5}{7}$ lbs. wt.; time of revolution $= \dfrac{\pi}{2}\sqrt{\dfrac{7}{2}}$ secs.

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